Lecture 14: Windowing

ECE 401: Signal and Image Analysis

University of Illinois

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1. Windowing
2. LCCDEs
3. Z Transform

Outline

Windowing Review

The following system implements a lowpass filter with a cutoff of $\omega_c = \frac{\pi}{6}$:

$$y[n] = \sum_{m=-17}^{17} x[n-m] \left(\frac{\sin(\pi m/6)}{\pi m} \right)$$

Unfortunately, this filter lets through a lot of energy in the stop-band. Design a filter, $h[m]$, with the same complexity (35 multiplications per output sample), but with a lot less stop-band ripple. Specify an $h[m]$ that accomplishes this goal.

Outline

DTFT Review

Remember the purpose of DTFT is to let us design filters with a carefully specified frequency response:

$$y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

$$X(\omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}$$

LCCDE

LCCDEs (linear constant coefficient difference equations) are a large important class of linear time-invariant systems. An LCCDE is defined by a set of feedforward coefficients b_m, $0 \leq m \leq M-1$, and a set of feedback coefficients a_n, $1 \leq n \leq N-1$:

$$y[n] = \sum_{m=0}^{M-1} b_m x[n-m] + \sum_{n=1}^{N-1} a_n y[n-m]$$

For example, an FIR filter is a sub-class of LCCDE, with $b_m = h[m]$:

$$y_{FIR}[n] = \sum_{m=0}^{M-1} h[m]x[n-m]$$

LCCDE: the Feedback Term

The feedback term in an LCCDE allows it to represent certain types of IIR (infinite impulse response) filters. For example, consider

$$y[n] = x[n] + 0.9y[n-1]$$

Notice that the impulse response of this system is

$$h[n] = (0.9)^n u[n]$$

LCCDE: Second Order Feedback

Or consider:

$$y[n] = 2a\sin(\theta)x[n-1] + 2a\cos(\theta)y[n-1] - a^2y[n-2]$$

The impulse response of this system can be calculated to be...

$$h[n] = \begin{cases} 0 & n=0 \\ 2a\sin(\theta) & n=1 \\ 4a^2\sin(\theta)\cos(\theta) = 2a^2\sin(2\theta) & n=2 \\ 4a^3\cos(\theta)\sin(2\theta) - 2a^3\sin(\theta) = 2a^3\sin(3\theta) & n=3 \\ \ldots & \ldots \\ 2a^n\sin(n\theta) & n\geq 0 \end{cases}$$

The above analysis is kinda clever, but much too hard to be done routinely. We need a better method to analyze feedback LCCDEs.

Analysis of LCCDEs using DTFT

Remember that the DTFT is linear. Therefore we can take the DTFT of both sides of this equation:

$$y[n] = \sum_{m=0}^{M-1} b_m x[n-m] + \sum_{n=1}^{N-1} a_n y[n-m]$$

In order to get:

$$Y(\omega) = \sum_{m=0}^{M-1} b_m \mathcal{F}\left\{x[n-m]\right\} + \sum_{n=1}^{N-1} a_n \mathcal{F}\left\{y[n-m]\right\}$$

where $\mathcal{F}\{x[n]\}$ means "the DTFT of $x[n]$". Obviously, the DTFT of $x[n]$ is $X(\omega)$. But what is $\mathcal{F}\{x[n-m]\}$?

Time-Shift Property of DTFT

Definition of the DTFT:

$$\mathcal{F}\left\{x[n-m]\right\} = \sum_{n=-\infty}^{\infty} x[n-m]e^{-j\omega n}$$

Define $k = n - m$, so

$$\mathcal{F}\left\{x[n-m]\right\} = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}e^{-j\omega m}$$

$$\mathcal{F}\left\{x[n-m]\right\} = e^{-j\omega m}X(\omega)$$

Analysis of LCCDEs using DTFT

Using the time-shift property of the DTFT, we can transform both sides of

$$y[n] = \sum_{m=0}^{M-1} b_m x[n-m] + \sum_{n=1}^{N-1} a_n y[n-m]$$

In order to get:

$$Y(\omega) = \sum_{m=0}^{M-1} b_m e^{-j\omega m} X(\omega) + \sum_{n=1}^{N-1} a_n e^{-j\omega n} Y(\omega)$$

$$Withalittlealgebra, weget \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{m=0}^{M-1} b_m e^{-j\omega m}}{1 - \sum_{m=0}^{N-1} a_m e^{-j\omega m}}$$

Analysis of LCCDEs using DTFT

But remember the convolution property of the DTFT: $Y(\omega) = H(\omega)X(\omega)$! So

$$H(\omega) = \frac{\sum_{m=0}^{M-1} b_m e^{-j\omega m}}{1 - \sum_{m=0}^{N-1} a_m e^{-j\omega m}}$$

Therefore

$$h[n] = \mathcal{F}^{-1}\left\{\frac{\sum_{m=0}^{M-1} b_m e^{-j\omega m}}{1 - \sum_{m=0}^{N-1} a_m e^{-j\omega m}}\right\}$$

where $\mathcal{F}^{-1}$ means "inverse Fourier transform of." In other words, if we knew how to inverse transform that thing, then we would know $h[n]$. Unfortunately, we don't know how to inverse transform that thing...and so we invent the "Z transform" to help us figure it out.

Outline

Really, the Z transform is just a way to write the DTFT using fewer letters. Instead of writing

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

we write

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

In particular, the time-shift property of the Z transform is exactly the same as the DTFT one, but with fewer letters:

$$\mathcal{F}\left\{x[n-m]\right\} = e^{-j\omega m}X(\omega), \quad \mathcal{Z}\left\{x[n-m]\right\} = z^{-m}X(z)$$

So instead of

$$H(\omega) = \frac{\sum_{m=0}^{M-1} b_m e^{-j\omega m}}{1 - \sum_{m=0}^{N-1} a_m e^{-j\omega m}}$$

we have

$$H(z) = \frac{\sum_{m=0}^{M-1} b_m z^{-m}}{1 - \sum_{m=0}^{N-1} a_m z^{-m}}$$

Z Transform of an Exponential Signal

Turning $e^{j\omega}$ into z is useful for a very small, but very important, set of signals. Specifically, it's useful for exponential signals. For example, suppose

$$x[n] = a^n u[n]$$

Then

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= \sum_{n=0}^{\infty} a^n z^{-n} \end{aligned}$$

$$X(z) = \frac{1}{1 - az^{-1}}, \quad \text{which means that} \quad X(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

Z Transform of Sine Wave

A particular kind of exponential signal that's really, really useful is the one called a "sine wave:"

$$x[n] = 2a^n \sin(\theta n) u[n]$$

Then

$$X(z) = \sum_{n=0}^{\infty} a^n \left(e^{j\theta n} - e^{-j\theta n}\right) z^{-n}$$

$$X(z) = \frac{1}{1 - ae^{j\theta}z^{-1}} - \frac{1}{1 - ae^{-j\theta}z^{-1}} = \frac{2a\sin(\theta)z^{-1}}{(1 - ae^{j\theta}z^{-1})(1 - ae^{-j\theta}z^{-1})}$$

...and you can kinda see why we like writing z instead of $e^{j\omega}$ all the time. It just saves space, really that's the main reason...

Z Transform of Cosiune

Another useful kind of exponential is the one called a "cosine:"

$$x[n] = 2a^n \cos(\theta n) u[n]$$

Then

$$X(z) = \sum_{n=0}^{\infty} a^n \left(e^{j\theta n} + e^{-j\theta n} \right) z^{-n}$$

$$X(z) = \frac{1}{1 - ae^{j\theta} z^{-1}} + \frac{1}{1 - ae^{-j\theta} z^{-1}} = \frac{2 - 2a\cos(\theta) z^{-1}}{(1 - ae^{j\theta} z^{-1})(1 - ae^{-j\theta} z^{-1})}$$

The Only Z Transform Pairs that Matter

$$x[n] = \delta[n] \leftrightarrow X(z) = 1$$

$$x[n] = a^n u[n] \leftrightarrow X(z) = \frac{1}{1 - az^{-1}}$$

$$x[n] = 2a^n \sin(\theta n) u[n] \leftrightarrow X(z) = \frac{2a\sin(\theta)z^{-1}}{(1 - ae^{j\theta}z^{-1})(1 - ae^{-j\theta}z^{-1})}$$

$$x[n] = 2a^n \cos(\theta n) u[n] \leftrightarrow X(z) = \frac{2 - 2a\cos(\theta)z^{-1}}{(1 - ae^{j\theta}z^{-1})(1 - ae^{-j\theta}z^{-1})}$$

Obviously, these transform pairs relate to the feedback LCCDEs we've solved so far. Let's explore the connection next time.