Lecture 16: Zeros and Poles

ECE 401: Signal and Image Analysis

University of Illinois

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Outline

Z Transform of a Unit Step

What's the Z transform of

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let's find out:

$$U(z) = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=-\infty}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}$$

This corresponds to

$$U(\omega) = \frac{1}{1 - e^{-j\omega}}$$

Notice that when $\omega = 0$, $|U(\omega)| = 1/(1-1) = 1/0 = \infty$. We say $U(z)$ has a **pole** at $\omega = 0$.

Filter with a Zero at DC

Consider the following filter:

$$y[n] = x[n] - x[n-1]$$

$$Y(z) = X(z) - z^{-1}X(z) = (1 - z^{-1})X(z)$$

$$H(z) = 1 - z^{-1}, \qquad H(\omega) = 1 - e^{-j\omega}$$

Notice that when $\omega = 0$, $|H(\omega)| = 0$. We say that $H(z)$ has a **zero** at $\omega = 0$.

Pole-Zero Cancellation

What happens when a pole meets a zero? Let's find out. Let's put $x[n] = u[n]$ into

$$y[n] = u[n] - u[n-1]$$

$$U(z) = \frac{1}{1 - z^{-1}}, \quad H(z) = (1 - z^{-1})$$

$$Y(z) = H(z)U(z) = \frac{1 - z^{-1}}{1 - z^{-1}} = 1$$

So $y[n]$ is the inverse Z-transform of $Y(z) = 1$, which is $y[n] = \delta[n]$. ...But we could have figured that out directly from the system equation!

A Filter with a Pole

Let's find the transfer function for this system:

$$y[n] = x[n] + y[n-1]$$

$$Y(z) = X(z) + z^{-1}Y(z)$$

$$Y(z)(1 - z^{-1}) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1}}$$

So this is a **transfer function** with a pole at $\omega = 0$!

Pole-Pole Coincidence

What happens when a pole in the input ($X(z)$) meets a pole in the transfer function ($H(z)$)? Let's use $u[n]$ as the input to this system:

$$y[n] = u[n] + y[n-1]$$

By plugging $u[n]$ into that equation directly, we discover that

$$y[n] = (n+1)u[n]$$

...which is almost certainly a **very bad thing**, because it grows without bound (we sometimes say it is "unbounded" or it "goes to infinity"). The Z transform is:

$$Y(z) = \frac{1}{1-z^{-1}}U(z) = \frac{1}{(1-z^{-1})^2}$$

So it has **two poles** at $\omega = 0$. We have learned that when $Y(z)$ has two poles at the same frequency, then $y[n]$ goes to infinity.

Summary so far: Poles and Zeros

- When $X(z)$ has a pole at some frequency, and $H(z)$ has a zero at the same frequency (or vice versa!!), then the pole and zero cancel.
- When $X(z)$ and $H(z)$ both have poles at the same frequency, then $y[n]$ goes to infinity.

Summary so far: Useful Z Transform Pairs

$y[n]$	$Y(z)$	Zeros	Poles
$\delta[n]-\delta[n-1]$	$(1-z^{-1})$	One at $\omega=0$	None
$\delta[n]$	1	None	None
$u[n]$	$\frac{1}{(1-z^{-1})}$	None	One at $\omega=0$
$(n+1)u[n]$	$\frac{1}{(1-z^{-1})^2}$	None	Two at $\omega=0$

Outline

Z Transform of an Exponential

What's the Z transform of

$$x[n] = e^{-Bn}u[n] = \begin{cases} e^{-Bn} & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let's find out:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} e^{-Bn}z^{-n} \frac{1}{1 - e^{-B}z^{-1}}$$

This corresponds to

$$U(\omega) = \frac{1}{1 - e^{-(B+j\omega)}}; \quad U(0) = \frac{1}{1 - e^{-B}}, \quad U(B) = \frac{1}{1 - e^{-B(1+j)}} \approx \frac{1}{\sqrt{2}} B$$

We say that this signal has a **pole of bandwidth B** at $\omega = 0$.

Filter with a Zero at DC

Consider the following filter:

$$y[n] = x[n] - e^{-B}x[n-1]$$

$$Y(z) = X(z) - e^{-B}z^{-1}X(z) = (1 - e^{-B}z^{-1})X(z)$$

$$H(z) = 1 - e^{-B}z^{-1}, \qquad H(\omega) = 1 - e^{-(B+j\omega)}$$

We say that $H(z)$ has a **zero of bandwidth B** at $\omega = 0$.

Pole-Zero Cancellation

What happens when a pole meets a zero? Let's find out. Let's put $x[n] = e^{-Bn}u[n]$ into

$$y[n] = x[n] - e^{-B}x[n-1]$$

$$U(z) = \frac{1}{1 - e^{-B}z^{-1}}, \quad H(z) = (1 - e^{-B}z^{-1})$$

$$Y(z) = H(z)U(z) = \frac{1 - e^{-B}z^{-1}}{1 - e^{-B}z^{-1}} = 1$$

So $y[n]$ is the inverse Z-transform of $Y(z) = 1$, which is $y[n] = \delta[n]$. ...But we could have figured that out directly from the system equation!

A Filter with a Pole

Let's find the transfer function for this system:

$$y[n] = x[n] + e^{-B} y[n-1]$$

$$Y(z) = X(z) + e^{-B} z^{-1} Y(z)$$

$$Y(z)(1 - e^{-B} z^{-1}) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - e^{-B} z^{-1}}$$

So this is a **transfer function** with a **pole of bandwidth B** at $\omega = 0$!

Pole-Pole Coincidence

What happens when a pole in the input ($X(z)$) meets a pole in the transfer function ($H(z)$)? Let's use $e^{-Bn}u[n]$ as the input to this system:

$$y[n] = u[n] + e^{-B}y[n-1]$$

By plugging $u[n]$ into that equation directly, we discover that

$$y[n] = (n+1)e^{-Bn}u[n]$$

This is OK, **as long as** $B > 0$. As long as $B > 0$, the output $y[n]$ will rise and then fall again. The Z transform is:

$$Y(z) = \frac{1}{1-e^{-B}z^{-1}}U(z) = \frac{1}{(1-e^{-B}z^{-1})^2}$$

So it has **two poles with bandwidth B** at $\omega = 0$. We have learned that when a pole has positive bandwidth, then $y[n]$ **doesn't** go to infinity.

Summary so far: Poles and Zeros

- When $X(z)$ has a pole at some frequency, and $H(z)$ has a zero at the same frequency **with the same bandwidth**, then the pole and zero cancel.
- When $X(z)$ and $H(z)$ both have poles at the same frequency but with positive bandwidth, then $y[n]$ doesn't go to infinity; it rises, then falls again.

Summary so far: Useful Z Transform Pairs

$y[n]$	$Y(z)$	Zeros	Poles
$\delta[n] - e^{-B}\delta[n-1]$	$(1 - e^{-B}z^{-1})$	$\omega = 0$, BW$=B$	None
$\delta[n]$	1	None	None
$e^{-Bn}u[n]$	$\frac{1}{(1-e^{-B}z^{-1})}$	None	$\omega = 0$, BW$=B$
$(n+1)u[n]$	$\frac{1}{(1-e^{-B}z^{-1})^2}$	None	Two at $\omega = 0$, BW$=B$

Outline

Z Transform of a Sinusoid

What's the Z transform of

$$x[n] = \cos(\theta n)u[n] = \begin{cases} \cos(\theta n) & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let's find out:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-j\theta n}z^{-n} + \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{j\theta n}z^{-n}$$

This corresponds to

$$U(\omega) = \frac{1}{2}\frac{1}{1-e^{-j(\omega+\theta)}} + \frac{1}{2}\frac{1}{1-e^{-j(\omega-\theta)}}$$

Notice that $|U(\omega)| = \frac{1}{0} = \infty$ when $\omega = \pm\theta$. We say that $U(z)$ has poles at $\omega = \pm\theta$

Filter with a Zero at DC

Consider the following filter:

$$y[n] = x[n] - e^{j\theta}x[n-1]$$

$$Y(z) = X(z) - e^{j\theta}z^{-1}X(z) = (1 - e^{j\theta}z^{-1})X(z)$$

$$H(z) = 1 - e^{j\theta}z^{-1}, \qquad H(\omega) = 1 - e^{-j(\omega-\theta))}$$

We say that $H(z)$ has a **zero** at $\omega = \theta$.

Pole-Zero Cancellation

What happens when a pole meets a zero? Let's find out. Let's put $x[n] = e^{j\theta n}u[n]$ into

$$y[n] = x[n] - e^{j\theta}x[n-1]$$

$$X(z) = \frac{1}{1 - e^{j\theta}z^{-1}}, \quad H(z) = (1 - e^{j\theta}z^{-1})$$

$$Y(z) = H(z)X(z) = \frac{1 - e^{j\theta}z^{-1}}{1 - e^{j\theta}z^{-1}} = 1$$

So $y[n]$ is the inverse Z-transform of $Y(z) = 1$, which is $y[n] = \delta[n]$. ...But we could have figured that out directly from the system equation!

A Filter with a Pole

Let's find the transfer function for this system:

$$y[n] = x[n] + e^{j\theta} y[n-1]$$

$$Y(z) = X(z) + e^{j\theta} z^{-1} Y(z)$$

$$Y(z)(1 - e^{j\theta} z^{-1}) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - e^{j\theta} z^{-1}}$$

So this is a **transfer function** with a **pole of zero bandwidth B** at $\omega = \theta$!

Pole-Pole Coincidence

What happens when a pole in the input ($X(z)$) meets a pole in the transfer function ($H(z)$)? Let's use $x[n] = e^{j\theta n}u[n]$ as the input to this system:

$$y[n] = x[n] + e^{j\theta}y[n-1]$$

By plugging $x[n]$ into that equation directly, we discover that

$$y[n] = (n+1)e^{j\theta n}u[n]$$

This is a **very bad thing**, because $|e^{j\theta n}| = 1$. Therefore the magnitude of $y[n]$ is $|y[n]| = (n+1)$, which goes to infinity. The Z transform is:

$$Y(z) = \frac{1}{1-e^{j\theta}z^{-1}}X(z) = \frac{1}{(1-e^{j\theta}z^{-1})^2}$$

So it has **two poles with zero bandwidth** at $\omega = \theta$. We have learned that when $Y(z)$ has two poles of zero bandwidth, at **any center frequency** $\omega = \theta$, then $y[n]$ goes to infinity.

Summary so far: Poles and Zeros

- When $X(z)$ has a pole at some frequency, and $H(z)$ has a zero at the same frequency, then the pole and zero cancel.
- When $X(z)$ and $H(z)$ both have poles at the same frequency with zero bandwidth, then $y[n]$ goes to infinity.

Summary so far: Useful Z Transform Pairs

$y[n]$	$Y(z)$	Zeros	Poles
$\delta[n] - e^{j\theta}\delta[n-1]$	$(1 - e^{j\theta}z^{-1})$	$\omega = \theta$, BW=0	None
$\delta[n]$	1	None	None
$e^{j\theta n}u[n]$	$\frac{1}{(1-e^{j\theta}z^{-1})}$	None	$\omega = \theta$, BW=0
$(n+1)e^{j\theta n}u[n]$	$\frac{1}{(1-e^{j\theta}z^{-1})^2}$	None	Two at $\omega = \theta$, BW=0

Outline

Narrowband Noise

Suppose our measurement, $x[n]$, includes a desired signal, $s[n]$, that has been corrupted by narrowband noise $v[n]$:

$$x[n] = s[n] + v[n]$$

Where $v[n]$ is a cosine at a **known frequency** $\omega = \theta$, but with unknown phase ϕ, and unknown amplitude A:

$$v[n] = A\cos(\theta n + \phi)$$

$$V(z) = \frac{A}{2}\left(\frac{e^{j\phi}}{1 - e^{j\theta}z^{-1}} + \frac{e^{-j\phi}}{1 - e^{-j\theta}z^{-1}}\right)$$

We call $v[n]$ a "narrowband" noise, because $V(z) = \infty$ at $\omega = \pm\theta$, and $V(z)$ is small or zero at all other frequencies.

Noise Removal

We want to design an LCCDE that will get rid of the noise. In other words, we want to find coefficients a_m and b_m so that

$$y[n] = \sum_{m=0}^{M-1} b_m x[n-m] + \sum_{m=0}^{N-1} a_m y[n-m]$$

and

$$y[n] \approx s[n]$$

Noise Removal

Let's put it in the Z transform domain:

$$Y(z) = H(z)X(z) = H(z)S(z) + H(z)V(z)$$

where

$$H(z) = \frac{\sum_{m=0}^{M-1} b_m z^{-m}}{1 - \sum_{m=0}^{N-1} a_m z^{-m}}$$

We want to design $H(z)$ so that:

- $h[n] * v[n] = 0$ for all $n > 0$. For example, we can design it so that $h[n] * v[n] = \delta[n]$ by using pole-zero cancellation.
- $H(z)S(z) \approx S(z)$.

Part One: the Zeros

First, let's find $V(z)$.

$$V(z) = \sum_{n=0}^{\infty} A\cos\left(\theta n + \phi\right) z^{-n}$$

$$= \frac{Ae^{j\phi}}{2}\sum_{n=0}^{\infty} e^{j\theta n} z^{-n} + \frac{Ae^{-j\phi}}{2}\sum_{n=0}^{\infty} e^{-j\theta n} z^{-n}$$

$$= \frac{Ae^{j\phi}/2}{1 - e^{j\theta}z^{-1}} + \frac{Ae^{-j\phi}/2}{1 - e^{-j\theta}z^{-1}}$$

Part One: the Zeros

$$V(z) = \frac{Ae^{j\phi}/2}{1 - e^{j\theta}z^{-1}} + \frac{Ae^{-j\phi}/2}{1 - e^{-j\theta}z^{-1}}$$

We can cancel these two poles by using zeros:

$$H(z) = \frac{(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1})}{\text{something}}$$

So that, right at the two frequencies $\omega = \pm\theta$, $H(z) = 0$, and therefore, right at those two noise frequencies, $Y(z) = 0$.

Part Two: The Poles

Recall that we want $H(z) = 0$ right at $\omega = \theta$, but at all other frequencies, we want $H(z)S(z) \approx S(z)$. In other words, at all frequencies **other** than $\omega = \theta$, we want $H(z) \approx 1$. This can be done by giving $H(z)$ a pair of poles at exactly the same frequency, but with small positive bandwidth:

$$H(z) = \frac{(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1})}{(1 - e^{-B+j\theta}z^{-1})(1 - e^{-B-j\theta}z^{-1})}$$

This has the following properties:

- Right at $\omega = \pm\theta$, the numerator ensures that $H(z) = 0$ exactly.
- At frequencies such that $|\omega - \theta| \gg B$, the numerator and denominator cancel each other out, so that $H(z) \approx 1$.

The LCCDE

Let's turn it into an LCCDE.

$$H(z) = \frac{(1-e^{j\theta}z^{-1})(1-e^{-j\theta}z^{-1})}{(1-e^{-B+j\theta}z^{-1})(1-e^{-B-j\theta}z^{-1})}$$

$$H(z) = \frac{1-2\cos\theta z^{-1}+z^{-2}}{1-2e^{-B}\cos\theta z^{-1}+e^{-2B}z^{-2}}$$

So the LCCDE is:

$$y[n] = x[n]-2\cos\theta x[n-1]+x[n-2]+2e^{-B}\cos\theta y[n-1]-e^{-2B}y[n-2]$$