Lecture 18: Stability

ECE 401: Signal and Image Analysis

University of Illinois

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Outline

BIBO Stability

A system is "BIBO Stable" (bounded-input-bounded-output) if and only if **every** bounded input yields a bounded output. In other words, a system is stable if and only if:

- For **EVERY** $x[n]$ such that
  - there is some finite A such that
  - $|x[n]| \leq A$ for every n,
- the corresponding $y[n]$ satisfies
  - $|y[n]| \leq B$ for every n,
  - for some finite B.

Example of a Stable System

Consider this system:

$$y[n] = 10,000x^2[n]$$

This system is stable because

$$|x[n]| \leq A \Leftarrow |y[n]| \leq 1000A^2$$

Since $1000A^2$ is a finite number, the system is stable.

System Gain

System Gain is defined to be the maximum possible ratio of output amplitude over input amplitude, as computed over all possible input signals:

$$G = \max_{x[n]} \frac{B}{A}$$

For example, for the system $y[n] = 1000x^2[n]$,

$$G = \max_{x[n]} \frac{1000A^2}{A} = 1000A$$

Key idea: an **unstable system** is one with **infinite gain**.

The Practical Use of Stability

Practical situation: you might have a system that can only represent signals less than some maximum amplitude, for example **fractional-arithmetic hardware** requires

$$\text{Typical hardware requirements: } |x[n]| \le 1, \quad |y[n]| \le 1$$

If G is finite, then you can implement the system on fractional-arithmetic hardware as follows:

$$\tilde{x}[n] = \frac{x[n]}{G}, \quad y[n] = \text{system}\left(\tilde{x}[n]\right)$$

Example of an Unstable System

Consider this system:

$$y[n] = x[n] * u[n]$$

This system is unstable. To prove it's unstable, we just need to find a bounded input (**any** bounded input) that generates an unbounded output. For example:

$$x[n] = u[n], \quad |x[n]| \leq 1$$

Produces

$$y[n] = (n+1)u[n]$$

which is unbounded.

Practical Consequences of Instability

In practice, instability is **hard to debug**, because it looks like your system is producing **silence** as the output. For example, suppose you have:

- $F_s = 44,100$ samples/second
- 16 bits/sample, so the largest possible sample value is $2^{15} = 32768$
- A system that generates $y[n] = x[n] + y[n-1]$
- The input $x[n] = 1000u[n]$.

...then in just 32 samples (less than one millisecond), the system will hit $y[n] = 32768$. After that, it will have $y[n] = 32768$, constant, forever. The system output will sound like perfect silence; you'll think your speakers have gone dead.

Outline

Use the Impulse Response to Test Stability

If the system is LTI, then you can use the impulse response to test for instability.

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

Suppose we have the requirement $|x[n-m]| \leq A$. Then the biggest possible output sample is achieved, at sample $y[n]$, if

$$x[n-m] = A\text{sign}\left(h[m]\right)$$

Then

$$y[n] = A \sum_{m=-\infty}^{\infty} |h[m]|$$

System Gain of an LTI System

The system gain of an LTI system is

$$G = \sum_{m=-\infty}^{\infty} |h[m]|$$

This worst-case input-output gain is achieved if, for **any value** of n, it turns out that

$$x[n-m] = A \text{sign}\left(h[m]\right)$$

Examples: Stable Systems

- All FIR systems are stable, as long as they have finite coefficients.
  - Suppose $|h[n]| \leq C$
  - Suppose $h[n]$ is only N samples long
  - Then the system gain is $G \leq NC$, which is finite, therefore the system is stable.
- Any IIR system with a decaying exponential impulse response is stable.
  - Suppose $h[n] = a^n \cos(\omega_0 n) u[n]$.
  - Then $|h[n]| < a^n$
  - If $a < 1$, then

$$\sum_{m=-\infty}^{\infty} |h[n]| < \sum_{m=0}^{\infty} a^n = \frac{1}{1-a}$$

Examples: Unstable Systems

- $h[m] = u[m]$ is unstable (worst-case input: $x[n] = u[n]$).
- $h[m] = \cos(\omega_0 m) u[m]$ is unstable (worst-case input: $x[n-m] = \text{sign}\left(\cos(\omega_0 m)\right)$)
- Even an ideal lowpass filter is unstable!!

$$\sum_{m=-\infty}^{\infty} \left| \frac{\sin(\omega_c n)}{\pi n} \right| = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \left| \frac{1}{\pi n} \right| = \infty$$

- ...and of course, $h[n] = a^n u[n]$ is unstable if $|a| > 1$.

Outline

LCCDE Systems

An LCCDE system has this transfer function:

$$H(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{1 - \sum_{m=1}^{N} a_m z^{-m}}$$

If $M \leq N$, then it has a partial fraction expansion:

$$H(z) = C_0 + \sum_{k=1}^{N} \frac{C_k}{1 - p_k z^{-1}}$$

So

$$h[n] = C_0 \delta[n] + \sum_{k=1}^{N} C_k p_k^n u[n]$$

LCCDE Systems

An LCCDE system has this impulse response:

$$h[n] = C_0\delta[n] + \sum_{k=1}^{N} C_k p_k^n u[n]$$

Each pole is a complex number, $p_k = a_k e^{j\theta_k}$. There are three possibilities:

1. $a_k < 1$. In this case, p_k^n is exponentially decaying, so this pole is stable.
2. $a_k > 1$. In this case, p_k^n is exponentially increasing, so this pole is unstable.
3. $a_k = 1$. In this case, $p_k^n + p_{k+1}^n = e^{j\theta_k n} + e^{-j\theta_k n} = 2\cos(\theta_k n)$, which is unstable: an input at the same frequency will cause $y[n]$ to be unbounded.

LCCDE System Stable IFF Poles Inside Unit Circle

An LCCDE system has this impulse response:

$$h[n] = C_0\delta[n] + \sum_{k=1}^{N} C_k p_k^n u[n]$$

- This system is stable if and only if $|p_k| < 1$ for all poles.
- We say that "an LCCDE is stable if and only if all of the poles are inside the unit circle."
- Notice it doesn't matter where the zeros are. Stability only requires the poles to be inside the unit circle, not the zeros.