Lecture 2: Phasors

ECE 401: Signal and Image Analysis

University of Illinois

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Outline

Sum of Cosines is a Cosine

$$A\cos(\omega n-\alpha)+B\sin(\omega n-\beta)=C\cos(\omega n-\gamma)$$

- When you add cosines at the same frequency (ω), the result is another cosine at that frequency.
- (sine is a type of cosine: $\sin(\omega n)=\cos(\omega n-\frac{\pi}{2})$)

What is C? What is Gamma?

$$A\cos(\omega n - \alpha) + B\sin(\omega n - \beta) = C\cos(\omega n - \gamma)$$

- You might know that you can find C and γ using trig identities, like $\cos a \cos b = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$. The problem with this method: no pictures.
- Goal of today's lecture: teach you a method that solves this problem using pictures.

Outline

Complex Number = Funny 2D Vector

A "complex number" is just a 2D vector with a funny multiplication rule.

$$x = (x_r, x_i), \quad y = (y_r, y_i)$$

$$x + y = (x_r + y_r, x_i + y_i)$$

$$xy = (x_r y_r - x_i y_i, x_r y_i + x_i y_r)$$

j = Square Root of -1

The "funny multiplication rule" happens to make sense if we pretend that $j=\sqrt{-1}$:

$$x = x_r + jx_i, \quad y = y_r + jy_i$$

$$x+y = (x_r+y_r) + j(x_i+y_i)$$

$$xy = x_ry_r + jx_ry_i + jx_iy_r + j^2x_iy_i$$

$$= (x_ry_r - x_iy_i) + j\left(x_ry_i + x_iy_r\right)$$

Magnitude and Phase

The "funny multiplication rule" is actually much easier to write in terms of magnitude and phase:

$$x = M_X e^{j\theta_x}, \quad y = M_y e^{j\theta_y}, \quad z = M_z e^{j\theta_x}$$

$$z = xy = M_X e^{j\theta_x} M_y e^{j\theta_y}$$

$$M_z = M_x M_y, \quad \theta_z = \theta_x + \theta_y$$

Euler's Identity

The "funny multiplication rule" results in Euler's identity:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$M_x e^{j\theta_x} = M_x\cos\theta_x + jM_x\sin\theta_x$$

$$x_r = M_x\cos\theta_x, \quad x_i = M_x\sin\theta_x$$

Quarter-Circles (Quadrature)

$$1 = \cos 0 + j \sin 0 = e^{j0}$$

$$j = \cos\left(\frac{\pi}{2}\right) + j \sin\left(\frac{\pi}{2}\right) = e^{j\pi/2}$$

$$(-1) = \cos(\pi) + j \sin(\pi) = e^{j\pi}$$

$$-j = \cos\left(-\frac{\pi}{2}\right) + j \sin\left(-\frac{\pi}{2}\right) = e^{-j\pi/2}$$

. . . we can add 2π to any of the above angles, and get the same result.

Magnitude and Phase

Conversely, to get back the magnitude and phase, we use

$$M_x = \sqrt{x_r^2 + x_i^2} = \sqrt{|x|^2} = \sqrt{xx^*}$$

...where x^* is a special number we made up just for this purpose, called "x conjugate:"

$$x^* = x_r - jx_i$$

The angle can be defined to be $-\pi < \theta \le \pi$, if we're careful about x_r.

$$\frac{x_i}{x_r} = \frac{\sin\theta_x}{\cos\theta_x} = \tan\theta_x = \tan(\theta_x \pm \pi) = \frac{\sin(\theta_x \pm \pi)}{\cos(\theta_x \pm \pi)} = \frac{-x_i}{-x_r}$$

$$\theta_x = \begin{cases} \text{atan}\left(\frac{x_i}{x_r}\right) & x_r > 0 \\ \text{atan}\left(\frac{x_i}{x_r}\right) \pm \pi & x_r < 0 \end{cases}$$

Outline

Euler's Identity

Phasors start with Euler's identity:

$$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

And then we turn it around:

$$\cos(\omega n) = \Re\left\{e^{j\omega n}\right\}$$

In the equation above, $\Re$ means "real part of"

Phasor

Pretend every cosine, and every sine, is the projection, into the real world, of a complex number called a PHASOR, times $e^{j\omega n}$

$$A\cos(\omega n - \alpha) = \Re\left\{Ae^{-j\alpha}e^{j\omega n}\right\}, \quad \text{PHASOR} = Ae^{-j\alpha}$$

$$B\sin(\omega n - \beta) = \Re\left\{-jBe^{-j\beta}e^{j\omega n}\right\} \quad \text{PHASOR} = -jBe^{-j\beta}$$

Phasor Trick

Here's the trick that makes this method worthwhile:

$$A\cos(\omega n-\alpha)+B\sin(\omega n-\beta)$$

$$=\Re\left\{Ae^{-j\alpha}e^{j\omega n}\right\}+\Re\left\{-jBe^{-j\beta}e^{j\omega n}\right\}$$

$$=\Re\left\{Ae^{-j\alpha}e^{j\omega n}-jBe^{-j\beta}e^{j\omega n}\right\}$$

$$=\Re\left\{\left(Ae^{-j\alpha}-jBe^{-j\beta}\right)e^{j\omega n}\right\}$$

. . . and. . .

$$C\cos(\omega n-\gamma)=\Re\left\{Ce^{-j\gamma}e^{j\omega n}\right\}$$

. . . so. . .

Phasor Trick

$$C\cos(\omega n-\gamma)=A\cos(\omega n-\alpha)+B\sin(\omega n-\beta)$$

...can be solved more easily by solving...

$$Ce^{-j\gamma}=Ae^{-j\alpha}-jBe^{-j\beta}$$

- Two different methods of solving. Actually, both require about the same amount of algebra, but...
- The bottom equation can be solved using a picture. The picture actually helps, a lot, in checking your solution.

Phasor Method w/o the Picture

It's much better to do this with the picture. But without the picture, here's how it's done:

$$Ce^{-j\gamma} = Ae^{-j\alpha} - jBe^{-j\beta}$$

$$= A\cos(-\alpha) + jA\sin(-\alpha) - j\left(B\cos(-\beta) + jB\sin(-\beta)\right)$$

$$= \left(A\cos(-\alpha) - j^2 B\sin(-\beta)\right) + j\left(A\sin(-\alpha) - B\cos(-\beta)\right)$$

$$= \left(A\cos(\alpha) - B\sin(\beta)\right) - j\left(A\sin(\alpha) + B\cos(\beta)\right)$$

$$C = \sqrt{(A\cos(\alpha) - B\sin(\beta))^2 + (A\sin(\alpha) + B\cos(\beta))^2}$$

$$\gamma = -\text{atan}\frac{-\left(A\sin(\alpha) + B\cos(\beta)\right)}{\left(A\cos(\alpha) - B\sin(\beta)\right)}$$

Example

$$z(t) = \cos\left(2\pi 0.01n - \frac{\pi}{4}\right) + \sin\left(2\pi 0.01n - \frac{\pi}{4}\right)$$

On-Board Practice

Can I have 3 volunteers to come try this one on the board? Thanks!

$$z(t) = \cos\left(0.26\pi n - \frac{\pi}{3}\right) + \sin\left(0.26\pi n - \frac{\pi}{6}\right)$$