Lecture 3: Spectrum

ECE 401: Signal and Image Analysis

University of Illinois

1/26/2017

Outline

On-Board Practice

Can I have 3 volunteers to come try this one on the board? Thanks!

$$z[n] = \cos\left(0.26\pi n - \frac{\pi}{3}\right) + \sin\left(0.26\pi n - \frac{\pi}{6}\right)$$

Find the phasors x and y, add them to find the phasor z, then convert it back to $z[n]$.
Hint: this one is easiest if you remember that the phasor of $\cos(\omega n)$ is $x = 1$, whereas the phasor of $\sin(\omega n)$ is $-j$.

Outline

Spectrum: Sum of Sinusoids

How do we represent the information in a signal like

$$z(t) = 5\cos(300\pi t) + 3\sin(500\pi t)$$

1. Complex spectrum (in linear units)
2. Power spectrum (Watts, or dB)
3. Energy spectrum (Joules, or dB)

Complex Spectrum

Complex spectrum is based on inverting Euler's identity:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

therefore

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

Complex Spectrum

For example

$$5\cos(300\pi t)+3\sin(500\pi t) = \frac{5}{2}e^{j300\pi t}+\frac{5}{2}e^{-j300\pi t}+\frac{3}{2j}e^{j500\pi t}-\frac{3}{2j}e^{-j500\pi t}$$

therefore

Ω (radians/sec)	F (Hz)	$X(\Omega)$ (Complex Spectrum)
-500π	-250	$-\frac{3}{2j} = 1.5j$
-300π	-150	$\frac{5}{2} = 2.5$
300π	150	$\frac{5}{2} = 2.5$
500π	250	$\frac{3}{2j} = -1.5j$

New concept: spectrum has content at **negative frequencies**. This is just a way of talking about sines vs. cosines, because $\sin(-x) = -\sin x$ but $\cos(-x) = \cos(x)$.

The "DC Term"

For example

$$7+5\cos(300\pi t)+3\sin(500\pi t) = 7e^{j0}+\frac{5}{2}e^{j300\pi t}+\frac{5}{2}e^{-j300\pi t}+\frac{3}{2j}e^{j500\pi t}-\frac{3}{2}$$

therefore

Ω (radians/sec)	F (Hz)	$X(\Omega)$ (Complex Spectrum)
-500π	-250	$-\frac{3}{2j} = 1.5j$
-300π	-150	$\frac{5}{2} = 2.5$
0	0	7
300π	150	$\frac{5}{2} = 2.5$
500π	250	$\frac{3}{2j} = -1.5j$

New concept: adding a constant is like adding a cosine at frequency $\Omega = 0$.

Outline

Power Spectrum

The power of any wave (sound, voltage, etc) is always proportional to the square of the wave. Acoustic wave: Watts $=$ Pascals^2/acoustic_ohms. Electric wave: Watts $=$ Volts^2/Ohms. And so on.
Ignore the constant, and focus on the square.

$$z(t) = A\cos(2\pi Ft - \theta)$$

$$P_z = \text{Average}\left(A^2\cos^2(2\pi Ft - \theta)\right)$$

$$= \text{Average}\left(A^2\left(\frac{1}{2} + \frac{1}{2}\cos(4\pi Ft - 2\theta)\right)\right)$$

$$= \frac{A^2}{2}$$

New concept: power of any sinusoid is independent of its phase.

Power Spectrum

The power of the sinusoid ($A^2/2$) gets divided between the positive-frequency half ($A^2/4$) and negative-frequency half ($A^2/4$), thus

$$z(t) = 7 + 5\cos(300\pi t) + 3\sin(500\pi t)$$

has the following power spectrum:

Ω (radians/sec)	F (Hz)	$\lvert X(\Omega)\rvert^2$ (Power Spectrum)
-500π	-250	$9/4$
-300π	-150	$25/4$
0	0	49
300π	150	$25/4$
500π	250	$9/4$

Parseval's Theorem

If $z(t)$ is periodic with any period T_0, then the average power can be computed in the time domain by averaging the square of the signal:

$$P_z = \frac{1}{0.02}\int_0^{0.02}\left(7 + 5\cos(300\pi t) + 3\sin(500\pi t)\right)^2 dt = 66$$

Or in the frequency domain by adding up the terms:

Ω (radians/sec)	F (Hz)	$\lvert X(\Omega)\rvert^2$ (Power Spectrum)
-500π	-250	$9/4$
-300π	-150	$25/4$
0	0	49
300π	150	$25/4$
500π	250	$9/4$

$$P_z = \frac{9+25+25+9}{4} + 49 = 66$$

Parseval's Theorem: Power is same in time domain or in frequency domain.

Decibels

Humans hear loudness roughly in proportion to the logarithm of power. The **Level** of a signal is $10\log_{10}|X(\Omega)|^2$:

Ω (radians/sec)	F (Hz)	$\|X(\Omega)\|^2$	$10\log_{10}\|X(\Omega)\|^2$ (dB)
-500π	-250	$9/4$	3.5dB
-300π	-150	$25/4$	8dB
0	0	49	17dB
300π	150	$25/4$	8dB
500π	250	$9/4$	3.5dB

New concept: the 150Hz component is 4.5dB "louder" (higher level) than the 250Hz component.

Energy Spectrum

The lab will use "energy spectrum," which is just the time integral of power (Joules = Watts × seconds). Energy only makes sense if you choose a total length of time, for example, if $T_0 = 0.02$ you could use

$$E_z = T_0 \times P_x = \int_0^{0.02} (7 + 5\cos(300\pi t) + 3\sin(500\pi t))^2 \, dt = 66$$

Ω (radians/sec)	F (Hz)	Power	Energy
-500π	-250	$9/4$	$0.02 \times 9/4$
-300π	-150	$25/4$	$0.02 \times 25/4$
0	0	49	0.02×49
300π	150	$25/4$	$0.02 \times 25/4$
500π	250	$9/4$	$0.02 \times 9/4$

$$E_z = 0.02 \times \left(\frac{9 + 25 + 25 + 9}{4} + 49 \right) = 0.02 \times 66$$

Outline

Amplitude Modulation

Suppose we take some "carrier wave" $x(t) = \cos(2000\pi t)$, and multiply it by a "modulating signal" $y(t) = \sin(100\pi t)$.

$$z(t) = x(t)y(t)$$

$$= \left(\frac{e^{j2000\pi t} + e^{-j2000\pi t}}{2}\right)\left(\frac{e^{j100\pi t} - e^{-j100\pi t}}{2j}\right)$$

$$= \frac{1}{4j}e^{j2100\pi t} - \frac{1}{4j}e^{j1900\pi t} + \frac{1}{4j}e^{-j1900\pi t} - \frac{1}{4j}e^{-j2100\pi t}$$

$$= \frac{1}{2}\sin(2100\pi t) + \frac{1}{2}\sin(1900\pi t)$$

Amplitude Modulation

In general,

$$z(t) = \cos\left(\Omega_1 t - \theta_1\right)\cos\left(\Omega_2 t - \theta_2\right)$$

is the same as

$$z(t) = \frac{1}{2}\cos\left((\Omega_1 + \Omega_2)t - (\theta_1 + \theta_2)\right) + \frac{1}{2}\cos\left((\Omega_1 - \Omega_2)t - (\theta_1 - \theta_2)\right)$$

Beat Tones

In fact, if you add together two pure tones very close together in frequency:

$$z(t) = \cos\Omega_1 t + \cos\Omega_2 t$$

People will hear it as an amplitude modulated tone:

$$z(t) = \cos(\frac{(\Omega_1 - \Omega_2)}{2}t)\cos(\frac{(\Omega_1 + \Omega_2)}{2}t)$$

Example: tuning a guitar string.