Lecture 4: Fourier Series

ECE 401: Signal and Image Analysis

University of Illinois

1/26/2017

Outline

1. Spectrum Review
2. Periodic Signals
3. Fourier Series

On-Board Practice

Can I have 3 volunteers to come try this one on the board? Thanks!

$$z[n] = 2 + 3\cos(880\pi t) + \sin(1760\pi t)$$

Plot the power spectrum, including positive, negative, and zero frequencies.

Outline

Periodic Signal

A periodic signal is one that repeats every T_0 seconds or every N_0 samples. T_0 or N_0 is called the "period." For example:

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{T_0}{2} \\ -1 & \frac{T_0}{2} \le t < T_0 \\ x(t - T_0) & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} 1 & 0 \le n < \frac{N_0}{2} \\ -1 & \frac{N_0}{2} \le n < N_0 \\ x[n - N_0] & \text{otherwise} \end{cases}$$

Fundamental Frequency

The "fundamental frequency" of $x(t)$ or $x[n]$ is the frequency of a cosine with the same period:

- $\Omega_0 = \frac{2\pi}{T_0}$ radians/second
- $F_0 = \frac{1}{T_0}$ Hertz (cycles/second)
- $\omega_0 = \frac{2\pi}{N_0}$ radians/sample
- $f_0 = \frac{1}{N_0}$ cycles/sample

Sums of Sinusoids

- A sum of sinusoids is periodic, with period equal to the least common multiple (LCM) of the nonzero periods.
- The fundamental frequency is the greatest common divisor (GCD) of the nonzero frequencies
- If the ratio of the two freqs is irrational, then their GCD is 0, their LSM is infinity, and the signal is not periodic.

$$x(t) = 2 + 2\cos(880\pi t) + \sin(1760\pi t), \quad \Omega_0 = 880\pi, \quad T_0 = \frac{1}{440}\text{ sec}$$

$$x[n] = \cos(0.04\pi n) + \sin(0.06\pi n), \quad \omega_0 = 0.02\pi, \quad N_0 = \frac{1}{0.01} = 100$$

Outline

Joseph Fourier, 1768-1830

Fourier Series

Fourier showed that **every** periodic signal is the sum of sinusoids. For example,

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{T_0}{2} \\ -1 & \frac{T_0}{2} \le t < T_0 \\ x(t-T_0) & \text{otherwise} \end{cases}$$

$$= \sum_{k=1,\ \ k \text{ odd}}^{\infty} \frac{4}{\pi k} \sin\left(\frac{2\pi k t}{T_0}\right)$$

Principle of Orthogonality

The trick: cosines are **orthogonal** when integrated over one period. Suppose we take two different cosines:

$$x(t) = e^{\frac{j2\pi kt}{T_0}}, \qquad y(t) = e^{\frac{j2\pi \ell t}{T_0}}$$

Multiply $x(t)$ times $y^*(t)$, and integrate them over one period:

$$\int_0^{T_0} x(t)y^*(t)dt = \int_0^{T_0} e^{\frac{j2\pi kt}{T_0}} e^{\frac{-j2\pi \ell t}{T_0}} dt$$

$$= \int_0^{T_0} e^{\frac{j2\pi(k-\ell)t}{T_0}} dt$$

$$= \int_0^{T_0} \left(\cos\left(\frac{2\pi(k-\ell)t}{T_0}\right) + j\sin\left(\frac{2\pi(k-\ell)t}{T_0}\right)\right) dt$$

$$= \begin{cases} \int_0^{T_0} 1dt = T_0 & k = \ell \\ 0 & k \neq \ell \end{cases}$$

Finding the Fourier Coefficients

In order to find the Fourier coefficients, we first **assume** that $x(t)$ has a Fourier series representation. For example, just **assume** that, for some set of complex numbers $X_0, X_1, \ldots,$ we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\frac{j2\pi kt}{T_0}}$$

Then we can find the ℓ^{th} coefficient, X_ℓ, by using the orthogonality principle:

$$\int_0^{T_0} x(t) e^{\frac{-j2\pi \ell t}{T_0}} dt = T_0 X_\ell$$

because all of the other terms ($k \neq \ell$) cancel out.

Fourier Series Summary

The method for computing the coefficients X_k from $x(t)$ is called the "forward transform:"

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\Omega_0 t}dt$$

The method for computing the signal $x(t)$ from the Fourier series coefficients X_k is called the "inverse transform:"

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}, \quad \Omega_0 = \frac{2\pi}{T_0}$$

Spectrum; Parseval's Theorem

Notice that, at frequency $k\Omega_0$, the signal has power $|X_k|^2$.

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}, \quad \Omega_0 = \frac{2\pi}{T_0}$$

The power spectrum is therefore

$$|X(k\Omega_0)|^2 = |X_k|^2$$

Parseval's theorem holds. The average power in the time domain is the same as the power in the frequency domain:

$$\frac{1}{T_0}\int_0^{T_0} x^2(t)dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Fourier Series Example

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{T_0}{2} \\ -1 & \frac{T_0}{2} \leq t < T_0 \\ x(t - T_0) & \text{otherwise} \end{cases}$$

$$X_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\Omega_0 t} dt$$

$$= \frac{1}{T_0} \int_0^{T_0/2} e^{-jk\Omega_0 t} dt - \frac{1}{T_0} \int_{T_0/2}^{T_0} e^{-jk\Omega_0 t} dt$$

$$= \frac{1}{-jk\Omega_0 T_0} \left[e^{-jk\Omega_0 t} \right]_0^{T_0/2} - \frac{1}{-jk\Omega_0 T_0} \left[e^{-jk\Omega_0 t} \right]_{T_0/2}^{T_0}$$

$$= \frac{\left((-1)^k - 1\right) - \left(1 - (-1)^k\right)}{-jk2\pi} = \begin{cases} \frac{4}{jk2\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Fourier Series: Other Forms

The Fourier series has three forms, called the **exponential**, **trigonometric**, and **compact trigonometric** forms. Your Ph.D. advisor might want you to know the other forms, and if so, you can look them up in the book. For this class, you only need to know the **exponential** form.

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}, \qquad \textbf{Exponential Form}$$

$$= \sum_{k=0}^{\infty} A_k \cos\left(k\Omega_0 t\right) + \sum_{k=1}^{\infty} B_k \sin\left(k\Omega_0 t\right), \qquad \textbf{Trigonometric Form}$$

$$= \sum_{k=0}^{\infty} C_k \cos\left(k\Omega_0 t + \theta_k\right), \qquad \textbf{Compact Trigonometric}$$