Lecture 6: Sampling

ECE 401: Signal and Image Analysis

University of Illinois

2/7/2017

Outline

On-Board Practice

Eastern Illinois is famous for two types of frogs. The Monotonal Illinois Frog (MIF) has a sinusoidal call at 50Hz:

$$b(t) = \sin(100\pi t)$$

The Bitonal Illinois Frog (BIF) has a call which is two sinusoids, at 49Hz and 51Hz:

$$m(t) = 0.5\sin(98\pi t) + 0.5\sin(102\pi t)$$

You have a signal; in order to determine if it's $b(t)$ or $m(t)$, you compute its spectrogram. How long does the spectrogram window need to be?

Outline

Definition of Sampling

"Sampling" means that you measure the signal (sound pressure, voltage, or whatever) once per $T = \frac{1}{F_s}$ seconds, and store the value in a computer.

$$x[n] = x(nT) = x(n/F_s)$$

2π Ambiguity in Phase

The problem with sampling is that $\cos(2\pi) = \cos(0)$. More generally,

$$e^{j\theta+2\pi k} = e^{j\theta}e^{j2\pi k} = e^{j\theta} \quad \text{for any integer } k$$

2π Ambiguity in Frequency

The 2π-ambiguity means that

$$\cos(\omega n) = \cos((\omega + 2\pi k)n) \quad \text{for any integer } k$$

- **Every sample** of the signal on the left is exactly the same as **every sample** of the signal on the right.
- The only way to tell them apart would be if you knew the function in the spaces between the samples, e.g., at $n = \frac{1}{2}$.
- But you don't. The integer values of n are all you get.
- So in this sense, the frequencies ω and $\omega \pm 2\pi$ are **exactly the same frequency**.

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Spectrum of a Discrete-Time Signal is Periodic

In order to represent the fact that ω and $\omega + 2\pi$ are actually the same angle, we usually say that **the spectrum of every discrete-time signal is "periodic" with period 2π:**

$$X(\omega) = X(\omega + 2\pi)$$

Spectrum of a Discrete-Time Signal is Periodic

For example,
$x[n] = \sin(0.137\pi n) = (\exp(0.137j\pi n) + \exp(-0.137j\pi n))/2j$ has the Fourier-series based power spectrum

$$|X_\omega|^2 = \begin{cases} \frac{1}{4} & \omega = 0.137\pi + 2\pi k \quad \text{for any integer } k \\ \frac{1}{4} & \omega = -0.137\pi + 2\pi k \quad \text{for any integer } k \\ 0 & \text{otherwise} \end{cases}$$

Often we will leave off the $2\pi k$, and assume that you understand it's still there. For example, the following equation has exactly the same meaning as the one above, because $X(\omega)$ is periodic:

$$|X_\omega|^2 = \begin{cases} \frac{1}{4} & \omega = 0.137\pi \\ \frac{1}{4} & \omega = -0.137\pi \\ 0 & \text{otherwise} \end{cases}$$

Principal Phase: Example

It usually makes most sense to define ω in the range $[-\pi, \pi]$. Thus, for example, an 875Hz tone

$$x(t) = \sin\left(1750\pi t\right)$$

when sampled at $F_s = 1000$samples/second produces

$$x[n] = \sin\left(\frac{1750\pi n}{1000}\right) = \sin\left(1.75\pi n\right)$$

$$= \sin\left(-0.25\pi n\right) = -\sin\left(0.25\pi n\right)$$

Principal Phase in Principle

So if the continuous-time frequency of a sine wave is Ω radians/second, then its discrete-time frequency can be defined in a few different ways:

Ambiguous definition: $\omega = \frac{\Omega}{F_s} \pm 2\pi k$ for any integer k

Unambiguous definition: $\omega = \text{mod}\left(\frac{\Omega}{F_s}, 2\pi\right)$

where the modulo operator is usually defined so that $-\pi < \omega \leq \pi$, but sometimes we'll wrap it around and define it so that $0 \leq \omega < 2\pi$.

Outline

Fourier Series

Remember that any periodic signal can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

So when sampled, it would be

$$x[n] = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 n}, \quad \omega_0 = \frac{\Omega_0}{F_s}$$

Except that formula is **INCORRECT** in the sense that, for large values of k, $k\omega_0 > \pi$! (I hate to call it incorrect; DECEPTIVE would be a better term. The signal really is $e^{jk\omega_0 n}$, but the signal is ALSO $e^{j(k\omega_0 - 2\pi)n}$).

Aliasing

Problem: the modulo operator "wraps around" all of the high-frequency components, back into the principal phase range:

$$x[n] = \sum_{k=-\infty}^{\infty} X_k e^{j\text{mod}(k\omega_0, 2\pi)n}$$

Since the summation is infinite, that means we have an infinite number of high-frequency harmonics wrapped around into the principal phase area. This means the signal is distorted.

Solution: Lowpass Filtering

The solution is to lowpass filter $x(t)$ (i.e., smooth it) before sampling it.
We lowpass filter it in order to zero-out any harmonics with frequency $k\Omega_0 > \pi F_s$, equivalent to $kF_0 > 0.5F_S$. Thus

$$y(t) = \text{LPF}(x(t)) = \sum_{-k_{max}}^{k_{max}} X_k e^{jk\Omega_0 t}$$

where $k_{max} = F_s/2F_0$. Then

$$y[n] = y(n/F_s) = \sum_{-k_{max}}^{k_{max}} X_k e^{jk\omega_0 n}$$

where we now have the guarantee that

$$-\pi \leq -k_{max}\omega_0 \quad \text{and} \quad k_{max}\omega_0 \leq \pi$$

Nyquist Theorem

It is possible to reconstruct a signal from its samples as long as its highest frequency component has a frequency $F < 0.5F_s$.