Lecture 7: Interpolation

ECE 401: Signal and Image Analysis

University of Illinois

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1. Sampling Review
2. Interpolation and Upsampling
3. Spectrum of Interpolated Signals

Outline

On-Board Practice

$x(t)$ is sampled at $F_{s,1} = 16,000$ samples/second, creating a signal $x[n]$. $x[n]$ is then played back through an ideal D/A at a different sampling rate, $F_{s,2} = 8,000$ samples/second, to create a signal $y(t)$. What is $y(t)$?

$$x(t) = 2 + 3\cos(2000\pi t) + \sin(20,000\pi t)$$

Outline

Interpolation and Upsampling

Today we'll learn upsampling, and four types of interpolation.

1. Upsampling: put zeros between the samples.
2. Piece-wise constant interpolation
3. Piece-wise linear interpolation
4. Piece-wise cubic spline interpolation
5. Sinc interpolation

Upsampling

Upsampling changes the sampling rate by inserting zeros. Suppose $x[n]$ is sampled at $F_{s,1}$, and we want to change the sampling rate to $F_{s,2} = MF_{s,1}$ for some integer M. Upsampling creates the signal $y[n]$:

$$y_{ups}[n] = \begin{cases} x[m] & n = mM \\ 0 & \text{otherwise} \end{cases}$$

Piece-Wise Constant

- Piece-wise constant interpolation creates

$$y_{PWC}[n] = x[m], \quad m = \text{int}\left(\frac{n}{M}\right)$$

  where the int operator takes the integer part.
- PWC interpolation can also be used as a kind of D/A, to create a continuous-time signal:

$$y_{PWC}(t) = x[m], \quad m = \text{int}\left(\frac{t}{T}\right)$$

  where $T = \frac{1}{F_s}$ is the sampling period of $x[m]$.
- A PWC signal is discontinuous once every M samples.

Piece-Wise Linear

- Piece-wise linear interpolation creates

$$y_{PWC}[n] = g\left(\frac{n-mM}{M}\right)x[m] + g\left(\frac{n-(m+1)M}{M}\right)x[m+1]$$

- PWL can also create a continuous-time signal:

$$y_{PWC}(t) = g\left(\frac{t-mT}{T}\right)x[m] + g\left(\frac{t-(m+1)T}{T}\right)x[m+1]$$

- PWL creates a **continuous** signal by using a continuous interpolation kernel:

$$g(t) = \max(0, 1-|t|)$$

Piece-Wise Cubic Spline

- Piece-wise cubic spline interpolation creates

$$y_{PWCS}[n] = \sum_{m=n/M-2}^{n/M+2} g\left(\frac{n-mM}{M}\right) x[m]$$

- PWCS can also create a continuous-time signal:

$$y_{PWCS}(t) = \sum_{m=n/M-2}^{n/M+2} g\left(\frac{t-mT}{T}\right) x[m]$$

- PWCS creates a **continuous** signal with **continuous** first derivatives. This is done by using an interpolation function that has continuous first derivatives:

$$g(t) = \begin{cases} 1-|t|^2 & 0 \le |t| \le 1 \\ 2(2-|t|)^3 - 2(2-|t|)^2 & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Sinc Interpolation

- Sinc interpolation creates

$$y_{SINC}[n] = \sum_{m=-\infty}^{\infty} g\left(\frac{n-mM}{M}\right) x[m]$$

- Sinc interpolation can also create a continuous-time signal:

$$y_{SINC}(t) = \sum_{m=-\infty}^{\infty} g\left(\frac{t-mT}{T}\right) x[m]$$

- Sinc interpolation creates a **continuous** signal with **all of its derivatives continuous**. It does this by using an interpolation function that has all continuous derivatives:

$$g(t) = \mathrm{sinc}(\pi t) \equiv \begin{cases} \frac{\sin(\pi t)}{\pi t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

Outline

Upsampling

Suppose a cosine with period T_0 is upsampled by a factor of M:

$$x[n] = \cos\left(2\pi n / T_0\right)$$

$$y[n] = \begin{cases} x[m] & n = mM \\ 0 & \text{otherwise} \end{cases}$$

Then $y[n]$ is periodic with period MT_0.

Fourier Series of an Upsampled Cosine

Since $y[n]$ has period MT_0, it can be written with a Fourier series:

$$y[n] = \sum_{k=0}^{MT_0-1} Y_k e^{jk\omega_0 n/M}, \quad \omega_0 = \frac{2\pi}{T_0}$$

The coefficients Y_k can be derived using Fourier series formula:

$$Y_k = \frac{1}{MT_0} \sum_{n=0}^{MT_0-1} y[n] e^{-jk\omega_0 n/M}$$

Since $y[n]$ is zero except at $n = mM$, we can write this as:

$$Y_k = \frac{1}{MT_0} \sum_{m=0}^{T_0-1} x[m] e^{-jk\omega_0 m}$$

$$= \begin{cases} \frac{1}{2M} & k\omega_0 = \pm\frac{2\pi}{T_0} + \ell 2\pi, \quad \text{any integer } \ell \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of an Upsampled Cosine

So

$$x[n] = \cos\left(2\pi n / T_0\right)$$

$$y[n] = \begin{cases} x[m] & n = mM \\ 0 & \text{otherwise} \end{cases}$$

Then $y[n]$ has the spectrum

$$Y_\omega = \begin{cases} \frac{1}{2M} & \omega = \pm\frac{2\pi}{MT_0} + \ell\frac{2\pi}{M}, \quad \text{for any integer } \ell \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of an Interpolated Cosine

- An interpolated cosine (PWC, PWL, or PWCS) has energy only at the frequencies where the upsampled cosine has energy, that is, at

$$\omega = \pm\frac{2\pi}{MT_0} + \ell\frac{2\pi}{M}$$

- The energy at the lowest harmonics ($\pm 2\pi/MT_0$) is nearly the same for interpolation as for upsampling.
- The better the interpolation, the more it damps out the high-frequency harmonics:

$$|Y_{PWCS,\omega}|^2 < |Y_{PWL,\omega}|^2 < |Y_{PWC,\omega}|^2 < |Y_{UPS,\omega}|^2, \quad \omega > \frac{2\pi}{MT_0}$$

Spectrum of Sinc Interpolation

Sinc interpolation completely eliminates the higher harmonics.

$$x[m] = \cos\left(\frac{2\pi m}{T_0}\right)$$

$$y[n] = \sum_{m=-\infty}^{\infty} \operatorname{sinc}\left(\frac{\pi(n-mM)}{M}\right) x[m]$$

Gives the following result exactly:

$$y[n] = \cos\left(\frac{2\pi n}{MT_0}\right)$$

It works in continuous time, too:

$$y(t) = \sum_{m=-\infty}^{\infty} \operatorname{sinc}\left(\frac{\pi(t-mT)}{T}\right) x[m] = \cos\left(\frac{2\pi t}{TT_0}\right)$$