# Lecture 9: Convolution

ECE 401: Signal and Image Analysis

University of Illinois

3/2/2017

1. Impulse Response Review
2. A Signal is Made of Impulses
3. Graphical Convolution
4. Properties of Convolution

# Outline

1. Impulse Response Review
2. A Signal is Made of Impulses
3. Graphical Convolution
4. Properties of Convolution

# Impulse Response

- The "impulse response" of a system, $h[n]$, is the output that it produces in response to an impulse input.

**Definition:** if and only if $x[n] = \delta[n]$ then $y[n] = h[n]$

- Given the system equation, you can find the impulse response just by feeding $x[n] = \delta[n]$ into the system.
- If the system is linear and time-invariant (terms we'll define later), then you can use the impulse response to find the output for **any** input, using a method called **convolution** that we'll learn in two weeks.
- For today, let's get some practice at finding the impulse response from the system equation.

# Impulse Response Review

Here is a system. What is its impulse response?

$$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$$

# Outline

1. Impulse Response Review
2. A Signal is Made of Impulses
3. Graphical Convolution
4. Properties of Convolution

# Shifted Impulse Response

- Suppose some system has impulse response $h[n]$.
- Suppose we put in the input $x[n] = \delta[n-3]$.
- Then the output will be $y[n] = h[n-3]$.
- Example:

$$h[n] = \begin{cases} 0.33333 & -1 \leq n \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

# Scaled Impulse Response

- Suppose some system has impulse response $h[n]$.
- Suppose we put in the input $x[n] = 15\delta[n-3]$.
- Then the output will be $y[n] = 15h[n-3]$.
- Example:

$$h[n] = \begin{cases} 0.33333 & -1 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$$

# Scaled Impulse Response

- Suppose some system has impulse response $h[n]$.
- Suppose we put in the input $x[n] = x[3]\delta[n-3]$.
- Then the output will be $y[n] = x[3]h[n-3]$.
- Example:

$$h[n] = \begin{cases} 0.33333 & -1 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$$

# Adding Impulse Responses

- Suppose some system has impulse response $h[n]$.
- Suppose we put in the input $x[n] = x[3]\delta[n-3] + x[4]\delta[n-4]$.
- Then the output will be $y[n] = x[3]h[n-3] + x[4]h[n-4]$.
- Example:

$$h[n] = \begin{cases} 0.33333 & -1 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$$

# Adding Impulse Responses

- Suppose some system has impulse response $h[n]$.
- Suppose we put in the input

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

- Then the output will be

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

- Example:

$$h[n] = \begin{cases} 0.33333 & -1 \leq n \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

# Adding Impulse Responses

- Suppose some system has impulse response $h[n]$.
- Suppose we put in the input

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

- Then the output will be

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

- Example:

$$h[n] = \begin{cases} 0.33333 & -1 \leq n \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

# Definition of Convolution

- Here's the trick: $x[n]$ is **always** equal to

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

- Therefore $y[n]$ is **always** equal to

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

- The above algorithm is called "convolution," and it has a special symbol:

$$y[n] = h[n] * x[n]$$

# Definition of Convolution

## Definition of Convolution

$$h[n] * x[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

# Outline

1. Impulse Response Review
2. A Signal is Made of Impulses
3. Graphical Convolution
4. Properties of Convolution

# Time Reversal

Suppose we try to plot $h[-m]$ as a function of $m$. The result looks like $h[m]$, but flipped around in time. Example:

$$h[m] = \begin{cases} 1 & 0 \le m \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$h[-m] = \begin{cases} 1 & -3 \le m \le 0 \\ 0 & \text{otherwise} \end{cases}$$

# Shifted Reversal

Suppose we try to plot $h[n-m]$ as a function of $m$. The result looks like $h[m]$, but flipped in time, and shifted by $n$. Example:

$$h[m] = \begin{cases} 1 & 0 \le m \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n-m] = \begin{cases} 1 & n-3 \le m \le n \\ 0 & \text{otherwise} \end{cases}$$

# Graphical Convolution

Suppose we're trying to calculate the function $y[n]$. The way we do it is:

- Plot $x[m]$ as a function of $m$.
- For each interesting value of $n$ (do as many as necessary, until we understand the whole pattern)
    - Plot $h[n-m]$ as a function of $m$.
    - Plot $x[m]h[n-m]$ as a function of $m$.
    - Compute $y[n] = \sum x[m]h[n-m]$

Example:

$$h[m] = \begin{cases} 1 & 0 \le m \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$x[m] = \begin{cases} 1 & 0 \le m \le 3 \\ 0 & \text{otherwise} \end{cases}$$

# Outline


1. Impulse Response Review
2. A Signal is Made of Impulses
3. Graphical Convolution
4. Properties of Convolution

# Properties of Convolution: Commutative

$$h[n] * x[n] = x[n] * h[n]$$

Putting it another way,

$$\sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

# Properties of Convolution: Shift

Suppose

$$y[n] = h[n] * x[n]$$

Then

$$y[n-n_0] = h[n-n_0] * x[n] = h[n] * x[n-n_0]$$

In other words, if you shift the input **or** the impulse response, then the output gets shifted. If you shift **both** the input and impulse response, then the output gets shifted twice:

$$y[n-2n_0] = h[n-n_0] * x[n-n_0]$$

# Properties of Convolution: Scaling

Suppose

$$y[n] = h[n] * x[n]$$

Then

$$Ay[n] = (Ah[n]) * x[n] = h[n] * (Ax[n])$$

In other words, if you scale the input **or** the impulse response, then the output gets scaled. If you scale **both** the input and impulse response, then the output gets scaled twice:

$$A^2 y[n] = (Ah[n]) * (Ax[n])$$

# Properties of Convolution: Time Reversal

Suppose

$$y[n] = h[n] * x[n]$$

Then

$$y[-n] = h[-n] * x[n] = h[n] * x[-n]$$

In other words, if you time-reverse either the input **or** the impulse response, then the output gets shifted. If you time-reverse **both** the input and impulse response, then the output gets time-reversed twice—which cancels out the time-reversal!!!

$$y[n] = h[-n] * x[-n]$$