Recurrent Neural Nets

ECE 417: Multimedia Signal Processing
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Outline

Basics of DSP: Filtering

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

$$Y(z) = H(z)X(z)$$

Finite Impulse Response (FIR)

$$y[n] = \sum_{m=0}^{N-1} h[m]x[n-m]$$

The coefficients, $h[m]$, are chosen in order to optimally position the $N-1$ zeros of the transfer function, r_k, defined according to:

$$H(z) = \sum_{m=0}^{N-1} h[m]z^{-m} = h[0]\prod_{k=1}^{N-1}\left(1 - r_k z^{-1}\right)$$

Infinite Impulse Response (IIR)

$$y[n] = \sum_{m=0}^{N-1} b_m x[n-m] + \sum_{m=1}^{M-1} a_m y[n-m]$$

The coefficients, b_m and a_m, are chosen in order to optimally position the $N-1$ zeros and $M-1$ poles of the transfer function, r_k and p_k, defined according to:

$$H(z) = \frac{\sum_{m=0}^{N-1} b_m z^{-m}}{1 - \sum_{m=1}^{M-1} a_m z^{-m}} = b_0 \frac{\prod_{k=1}^{N-1}\left(1 - r_k z^{-1}\right)}{\prod_{k=1}^{M-1}\left(1 - p_k z^{-1}\right)}$$

Outline

Convolutional Neural Net = Nonlinear(FIR)

$$y[n] = g\left(\sum_{m=0}^{N-1} h[m]x[n-m]\right)$$

The coefficients, $h[m]$, are chosen to minimize some kind of error. For example, suppose that the goal is to make $y[n]$ resemble a target signal $t[n]$; then we might use

$$E = \frac{1}{2}\sum_{n=0}^{N}(y[n]-t[n])^2$$

and choose

$$h[n] \leftarrow h[n] - \eta\frac{dE}{dh[n]}$$

Recurrent Neural Net (RNN) = Nonlinear(IIR)

$$y[n] = g\left(x[n] + \sum_{m=1}^{M-1} a_m y[n-m]\right)$$

The coefficients, a_m, are chosen to minimize the error. For example, suppose that the goal is to make $y[n]$ resemble a target signal $t[n]$; then we might use

$$E = \frac{1}{2}\sum_{n=0}^{N}\left(y[n] - t[n]\right)^2$$

and choose

$$a_m \leftarrow a_m - \eta\frac{dE}{da_m}$$

Outline

Review: Excitation and Activation

- The **activation** of a hidden node is the output of the nonlinearity (for this reason, the nonlinearity is sometimes called the **activation function**). For example, in a fully-connected network with outputs z_l, weights $\vec{v}$, bias v_0, nonlinearity $g()$, and hidden node activations $\vec{y}$, the activation of the l^{th} output node is

$$z_l = g\left(v_{l0} + \sum_{k=1}^{p} v_{lk} y_k\right)$$

- The **excitation** of a hidden node is the input of the nonlinearity. For example, the excitation of the node above is

$$e_l = v_{l0} + \sum_{k=1}^{p} v_{lk} y_k$$

Backprop = Derivative w.r.t. Excitation

- The **excitation** of a hidden node is the input of the nonlinearity. For example, the excitation of the node above is

$$e_l = v_{l0} + \sum_{k=1}^{p} v_{lk} y_k$$

- The gradient of the error w.r.t. the weight is

$$\frac{dE}{dv_{lk}} = \epsilon_l y_k$$

where ϵ_l is the derivative of the error w.r.t. the l^{th} **excitation**:

$$\epsilon_l = \frac{dE}{de_l}$$

Backprop for Fully-Connected Network

Suppose we have a fully-connected network, with inputs $\vec{x}$, weight matrices U and V, nonlinearities $g()$ and $h()$, and output z:

$$\begin{aligned} e_k &= u_{k0} + \sum_j u_{kj} x_j \\ y_k &= g\left(e_k\right) \\ e_l &= v_{l0} + \sum_k v_{lk} y_k \\ z_l &= h\left(e_l\right) \end{aligned}$$

Then the back-prop gradients are the derivatives of E with respect to the **excitations** at each node:

$$\begin{aligned} \epsilon_l &= \frac{dE}{de_l} \\ \delta_k &= \frac{dE}{de_k} \end{aligned}$$

Back-Prop in a CNN

Suppose we have a convolutional neural net, defined by

$$e[n] = \sum_{m=0}^{N-1} h[m]x[n-m]$$

$$y[n] = g\left(e[n]\right)$$

then

$$\frac{dE}{dh[m]} = \sum_n \delta[n]x[n-m]$$

where $\delta[n]$ is the back-prop gradient, defined by

$$\delta[n] = \frac{dE}{de[n]}$$

Back-Prop in an RNN

Suppose we have a recurrent neural net, defined by

$$e[n] = x[n] + \sum_{m=1}^{M-1} a_m y[n-m]$$
$$y[n] = g\left(e[n]\right)$$

then

$$\frac{dE}{da_m} = \sum_n \delta[n] y[n-m]$$

where $y[n-m]$ is calculated by forward-propagation, and then $\delta[n]$ is calculated by back-propagation as

$$\delta[n] = \frac{dE}{de[n]}$$

Outline

Partial vs. Full Derivatives

For example, suppose we want $y[n]$ to be as close as possible to some target signal $t[n]$:

$$E = \frac{1}{2}\sum_n (y[n] - t[n])^2$$

Notice that E depends on $y[n]$ in many different ways:

$$\frac{dE}{dy[n]} = \frac{\partial E}{\partial y[n]} + \frac{dE}{dy[n+1]}\frac{\partial y[n+1]}{\partial y[n]} + \frac{dE}{dy[n+2]}\frac{\partial y[n+2]}{\partial y[n]} + \ldots$$

Partial vs. Full Derivatives

In general,

$$\frac{dE}{dy[n]} = \frac{\partial E}{\partial y[n]} + \sum_{m=1}^{\infty} \frac{dE}{dy[n+m]} \frac{\partial y[n+m]}{\partial y[n]}$$

where

- $\frac{dE}{dy[n]}$ is the total derivative, and includes all of the different ways in which E depends on $y[n]$.
- $\frac{\partial y[n+m]}{\partial y[n]}$ is the partial derivative, i.e., the change in $y[n+m]$ per unit change in $y[n]$ if all of the other variables (all other values of $y[n+k]$) are held constant.

Partial vs. Full Derivatives

So for example, if

$$E = \frac{1}{2}\sum_n \left(y[n] - t[n]\right)^2$$

then the partial derivative of E w.r.t. $y[n]$ is

$$\frac{\partial E}{\partial y[n]} = y[n] - t[n]$$

and the total derivative of E w.r.t. $y[n]$ is

$$\frac{dE}{dy[n]} = \left(y[n] - t[n]\right) + \sum_{m=1}^{\infty} \frac{dE}{dy[n+m]} \frac{\partial y[n+m]}{\partial y[n]}$$

Partial vs. Full Derivatives

So for example, if

$$y[n] = g\left(x[n] + \sum_{m=1}^{M-1} a_m y[n-m]\right)$$

then the partial derivative of $y[n+k]$ w.r.t. $y[n]$ is

$$\frac{\partial y[n+k]}{\partial y[n]} = a_k \dot{g}\left(x[n+k] + \sum_{m=1}^{M-1} a_m y[n+k-m]\right)$$

where $\dot{g}(x) = \frac{dg}{dx}$ is the derivative of the nonlinearity. The total derivative of $y[n+k]$ w.r.t. $y[n]$ is

$$\frac{dy[n+k]}{dy[n]} = \frac{\partial y[n+k]}{\partial y[n]} + \sum_{j=1}^{k-1} \frac{dy[n+k]}{dy[n+j]} \frac{\partial y[n+j]}{\partial y[n]}$$

Synchronous Backprop vs. BPTT

The basic idea of back-prop-through-time is divide-and-conquer.

1. **Synchronous Backprop:** First, calculate the **partial derivative** of E w.r.t. the excitation $e[n]$ at time n, assuming that all other time steps are held constant.

$$\epsilon[n] = \frac{\partial E}{\partial e[n]}$$

2. **Back-prop through time:** Second, iterate backward through time to calculate the **total derivative**

$$\delta[n] = \frac{dE}{de[n]}$$

Synchronous Backprop in an RNN

Suppose we have a recurrent neural net, defined by

$$e[n] = x[n] + \sum_{m=1}^{M-1} a_m y[n-m]$$
$$y[n] = g\left(e[n]\right)$$
$$E = \frac{1}{2}\sum_n \left(y[n] - t[n]\right)^2$$

then

$$\epsilon[n] = \frac{\partial E}{\partial e[n]} = \left(y[n] - t[n]\right)\dot{g}\left(e[n]\right)$$

where $\dot{g}(x) = \frac{dg}{dx}$ is the derivative of the nonlinearity.

Back-Prop Through Time (BPTT)

Suppose we have a recurrent neural net, defined by

$$e[n] = x[n] + \sum_{m=1}^{M-1} a_m y[n-m]$$

$$y[n] = g\left(e[n]\right)$$

$$E = \frac{1}{2}\sum_n \left(y[n] - t[n]\right)^2$$

then

$$\delta[n] = \frac{dE}{de[n]}$$

$$= \frac{\partial E}{\partial e[n]} + \sum_{m=1}^{\infty} \frac{dE}{de[n+m]} \frac{\partial e[n+m]}{\partial e[n]}$$

$$= \epsilon[n] + \sum_{m=1}^{M-1} \delta[n+m]\dot{g}\left(e[n+m]\right) a_m$$

Outline

Vanishing/Exploding Gradient

- The "vanishing gradient" problem refers to the tendency of $\frac{dy[n+m]}{de[n]}$ to disappear, exponentially, when m is large.
- The "exploding gradient" problem refers to the tendency of $\frac{dy[n+m]}{de[n]}$ to explode toward infinity, exponentially, when m is large.
- If the largest feedback coefficient is $|a| > 1$, then you get exploding gradient. If not, you get vanishing gradient.

Example: Vanishing Gradient

Suppose that we have a very simple RNN:

$$y[n] = bx[n] + ay[n-1]$$

Suppose that $x[n]$ is only nonzero at time 0:

$$x[0] = x_0, \text{ and } x[n] = 0 \ \forall \ n \neq 0$$

Suppose that, instead of measuring $x[0]$ directly, we are only allowed to measure the output of the RNN m time-steps later. In order to encourage the neural net to learn $a \approx 1$, we might penalize any difference between $y[m]$ and x_0, thus:

$$E = \frac{1}{2}\left(y[m] - x_0\right)^2$$

Example: Vanishing Gradient

Now, how do we perform gradient update of the weights? If

$$y[n] = bx[n] + ay[n-1]$$

then

$$\frac{dE}{db} = \sum_n \left(\frac{dE}{dy[n]}\right) x[n]$$
$$= \left(\frac{dE}{dy[0]}\right) x[0]$$

But the error is defined as

$$E = \frac{1}{2}\left(y[m] - x_0\right)^2$$

so

$$\frac{dE}{dy[0]} = a\frac{dE}{dy[1]} = a^2\frac{dE}{dy[2]} = \ldots$$
$$= a^m\left(y[m] - x_0\right)$$

Example: Vanishing Gradient

So we find out that the gradient, w.r.t. the coefficient b, is either exponentially small, or exponentially large, depending on whether $|a| < 1$ or $|a| > 1$:

$$\frac{dE}{db} = x_0 \left(y[m] - x_0\right) a^m$$

In other words, if our application requires the neural net to wait m time steps before generating its output, then the gradient is exponentially smaller, and therefore training the neural net is exponentially harder.

Exponential Decay

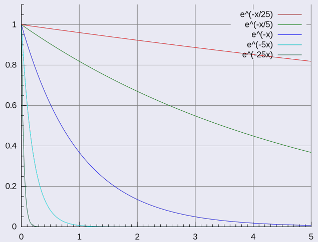


Image credit: PeterQ, Wikipedia

Outline

Gated Recurrent Units (GRU)

Gated recurrent units solve the vanishing gradient problem by making the feedback coefficient, $f[n]$, a sigmoidal function of the inputs. When the input causes $f[n] \approx 1$, then the recurrent unit remembers its own past, with no forgetting (no vanishing gradient). When the input causes $f[n] \approx 0$, then the recurrent unit immediately forgets all of the past.

$$y[n] = i[n]x[n] + f[n]y[n-1]$$

where the input and forget gates depend on $x[n]$ and $y[n]$, as

$$i[n] = \sigma\left(b_i x[n] + a_i y[n-1]\right) \in (0,1)$$
$$f[n] = \sigma\left(b_m x[n] + a_f y[n-1]\right) \in (0,1)$$

How does GRU work? Example

For example, suppose that the inputs just coincidentally have values that cause the following gate behavior:

$$i[n] = \begin{cases} 1 & n = n_0 \\ 0 & \text{otherwise} \end{cases}$$

$$f[n] = \begin{cases} 0 & n = n_0 \\ 1 & \text{otherwise} \end{cases}$$

$$y[n] = i[n]x[n] + f[n]y[n-1]$$

Then $y[N] = y[N-1] = \ldots = y[n_0] = x[n_0]$, memorized! And therefore

$$\frac{\partial y[N]}{\partial x[n]} = \begin{cases} 1 & n = n_0 \\ 0 & \text{otherwise} \end{cases}$$

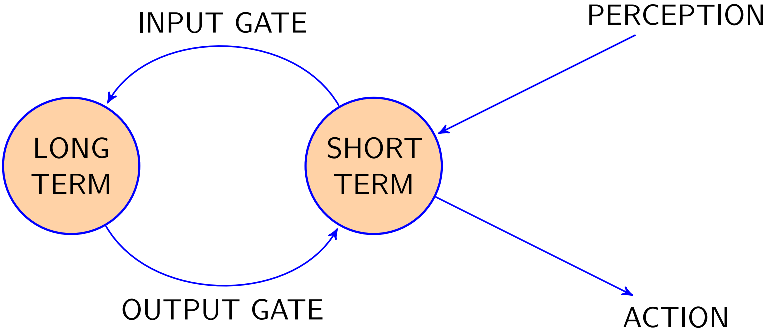
Training the Gates

$$
\begin{aligned}
y[n] &= i[n]x[n] + f[n]y[n-1] \\
i[n] &= \sigma\left(b_i x[n] + a_i y[n-1]\right) \in (0,1) \\
f[n] &= \sigma\left(b_m x[n] + a_f y[n-1]\right) \in (0,1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E}{\partial b_i} &= \sum_{n=0}^{N} \frac{\partial E}{\partial y[n]} \frac{\partial y[n]}{\partial i[n]} \frac{\partial i[n]}{\partial b_i} \\
&= \sum_{n=0}^{N} \delta[n] x[n] \frac{\partial i[n]}{\partial b_i}
\end{aligned}
$$

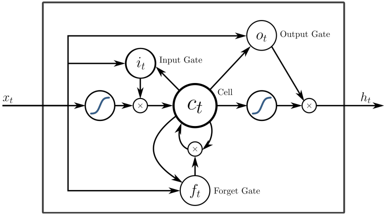
Outline

Characterizing Human Memory



$$\Pr\{\text{remember}\} = p_{LTM} e^{-t/T_{LTM}} + (1 - p_{LTM}) e^{-t/T_{STM}}$$

Neural Network Model: LSTM



$$
\begin{aligned}
i[n] &= \text{input gate} = \sigma(b_i x[n] + a_i c[n-1]) \\
o[n] &= \text{output gate} = \sigma(b_o x[n] + a_o c[n-1]) \\
f[n] &= \text{forget gate} = \sigma(b_f x[n] + a_f c[n-1]) \\
c[n] &= \text{memory cell}
\end{aligned}
$$

$$
\begin{aligned}
y[n] &= o[n]c[n] \\
c[n] &= f[n]c[n-1] + i[n]g\left(b_c x[n] + a_c c[n-1]\right)
\end{aligned}
$$

Outline

- TDNN is a one-dimensional ConvNet, the nonlinear version of an FIR filter. Coefficients are shared across time steps.
- RNN is the nonlinear version of an IIR filter. Coefficients are shared across time steps. Error is back-propagated from every output time step to every input time step.
- Vanishing gradient problem: the memory of an RNN decays exponentially.
- Solution: GRU
- An LSTM is a GRU with one more gate, allowing it to decide when to output information from LTM back to STM.