Derivation of the Matched Filter

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September 26, 2019

The signal detection problem can be stated as follows.

H0: $x[n] = v[n]$

H1: $x[n] = s[n] + v[n]$

where $s[n]$ is known, and $v[n]$ is zero-mean, unit variance Gaussian white noise.

To distinguish H0 versus H1, we want to design a filter $h[n]$ with the following properties:

- Maximize $y[0] = \sum_{m=-\infty}^{\infty} s[-m]h[m]$
- Subject to the constraint that $\sum_{m=-\infty}^{\infty} h^2[m] = 1$

It was specified in lecture that the problem, as stated above, is solved by $h[n] = \frac{s[-n]}{\sqrt{\sum_{m=-\infty}^{\infty} s^2[m]}}$. This is a sufficiently important result that everybody should know it.

The derivation is optional background knowledge (not required for an exam), but for those of you who are interested, it goes like this. Maximizing $y[0]$ subject to its constraint can be accomplished by maximizing the following Lagrangian, and then choosing λ so that the constraint is satisfied:

$$\mathcal{L} = \left(\sum_{m=-\infty}^{\infty} s[-m]h[m] \right) - \lambda \left(\sum_{m=-\infty}^{\infty} h^2[m] - 1 \right)$$

Differentiating $\mathcal{L}$ yields

$$\frac{\partial L}{\partial h[k]} = s[-k] - 2\lambda h[k]$$

Setting $\frac{\partial L}{\partial h[k]} = 0$ and solving, we obtain

$$h[k] = \frac{1}{2\lambda} s[-k]$$

In order to satisfy the constraint, we need to set

$$\lambda = \frac{1}{2} \sqrt{\sum_{m=-\infty}^{\infty} s^2[m]}$$

Therefore the filter $h[n]$ that maximizes $y[0]$ subject to the constraint is given by

$$h[n] = \frac{s[-n]}{\sqrt{\sum_{m=-\infty}^{\infty} s^2[m]}}$$