Lecture 5: Short-Time Fourier Transform and Filterbanks

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ECE 417: Multimedia Signal Processing, Fall 2020

Outline

Power Spectrum

The DFT power spectrum of a signal is defined to be $R[k] = \frac{1}{N}|X[k]|^2$. This is useful because the signal power is

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N}\sum_{k=0}^{N-1} R[k]$$

Similary, the DTFT power spectrum of a signal of length N can be defined to be $R(\omega) = \frac{1}{N}|X(\omega)|^2$, because the signal power is

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} R(\omega)d\omega$$

In this class we will almost never use the power spectrum of an infinite length signal, but if we need it, it can be defined as

$$R(\omega) = \lim_{N\rightarrow\infty} \frac{1}{N}\left|\sum_{n=-(N-1)/2}^{(N-1)/2} x[n]e^{-j\omega n}\right|^2$$

Autocorrelation

The power spectrum of a finite-length signal of length N is

$$R(\omega) = \frac{1}{N}|X(\omega)|^2$$

Its inverse Fourier transform is the autocorrelation,

$$r[n] = \frac{1}{N}x[n] * x[-n] = \frac{1}{N}\sum_{m=-\infty}^{\infty} x[m]x[m-n]$$

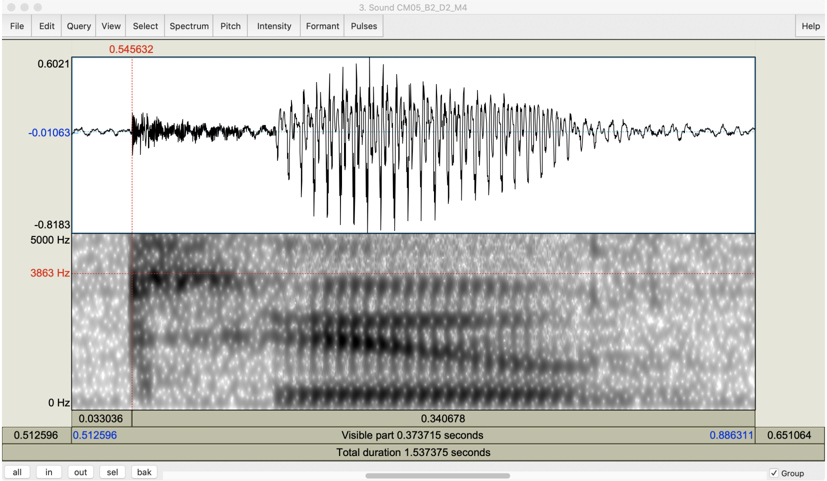
Or, if $x[n]$ is infinite-length, we can write

$$r[n] = \lim_{N\to\infty}\frac{1}{N}\sum_{m=-(N-1)/2}^{(N-1)/2} x[m]x[m-n]$$

This relationship, $r[n] \leftrightarrow R(\omega)$, is called Wiener's theorem, named after Norbert Wiener, the inventor of cybernetics.

Outline

Spectrogram $= 20\log_{10}|\text{Short Time Fourier Transform}|$



Short Time Fourier Transform

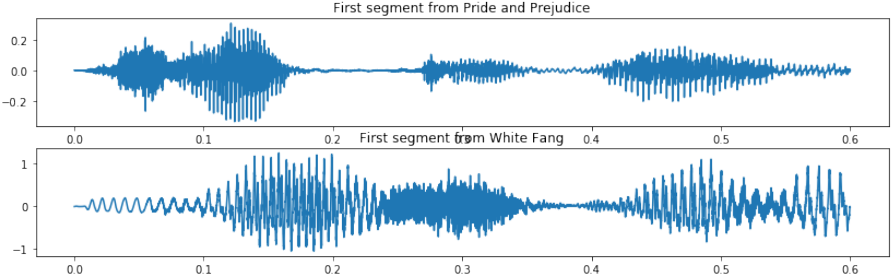
The short-time Fourier Transform (STFT) is the Fourier transform of a short part of the signal. We write either $X(\omega_k, m)$ of $X[k, m]$ to mean:

- The DFT of the short part of the signal that starts at sample m,
- windowed by a window of length less than or equal to N samples,
- evaluated at frequency $\omega_k = \frac{2\pi k}{N}$.

The next several slides will go through this procedure in detail, then I'll summarize.

Step #1: Chop out part of the signal

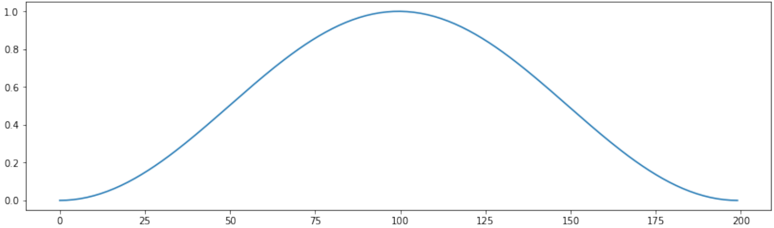
First, we just chop out the part of the signal starting at sample m. Here are examples from Librivox readings of *White Fang* and *Pride and Prejudice*:



Step #2: Window the signal

Second, we window the signal. A window with good spectral properties is the Hamming window:

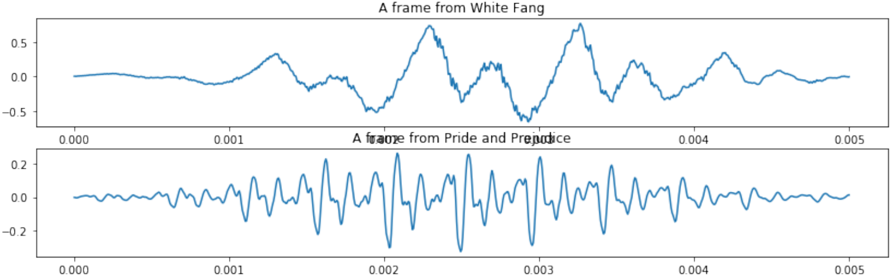
$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) & 0 \le n \le N \\ 0 & \text{otherwise} \end{cases}$$



Step #2: Window the signal

Here is the windowed signals, which is nonzero for $0 \leq n - m \leq (N-1)$:

$$x[n,m] = w[n-m]x[n]$$

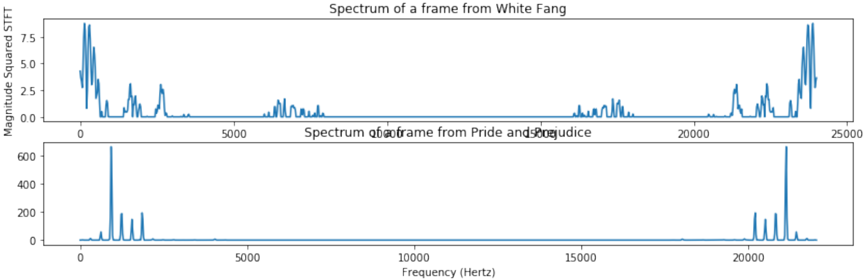


Step #3: Fourier Transform

Finally, we compute the DFT:

$$X[k,m] = \sum_{n=m}^{m+(N-1)} w[n-m]x[n]e^{-j2\pi k(n-m)/N}$$

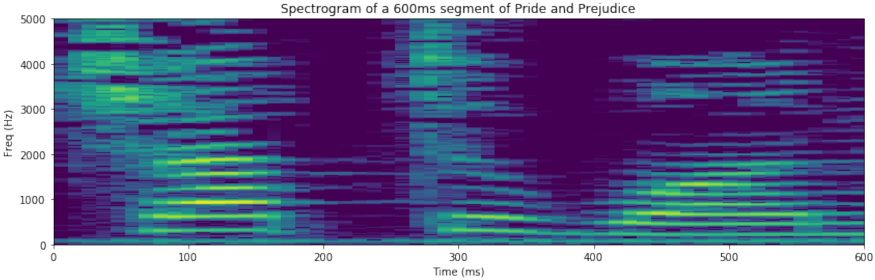
Here it is, plotted as a function of k:



Spectrogram = $20\log_{10}$ |Short Time Fourier Transform|

$$20\log_{10}|X[k,m]| = 20\log_{10}\left|\sum_n w[n-m]x[n]e^{-j2\pi k(n-m)/N}\right|$$

Here it is, plotted as an image, with k =row index, m =column index.



Putting it all together: STFT

The STFT, then, is defined as

$$X[k,m] = \sum_n w[n-m]x[n]e^{-j\omega_k(n-m)}, \quad \omega_k = \frac{2\pi k}{N}$$

which we can also write as

$$X[k,m] = \text{DFT}\left\{w[n]x[n+m]\right\}$$

Outline

STFT as a bank of analysis filters

The STFT is defined as:

$$X[k,m] = \sum_{n=m}^{m+(N-1)} w[n-m]x[n]e^{-j\omega_k(n-m)}$$

which we can also write as

$$X[k,m] = x[m] * h_k[-m]$$

where

$$h_k[m] = w[m]e^{j\omega_k m}$$

The frequency response of this filter is just the window DTFT, $W(\omega)$, shifted up to ω_k:

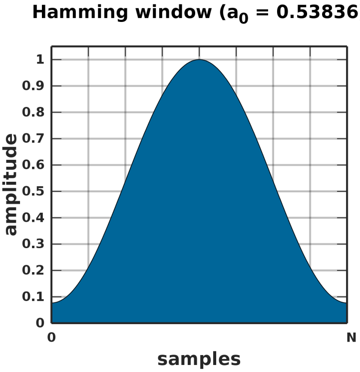
$$H_k(\omega) = W\left(\omega - \omega_k\right)$$

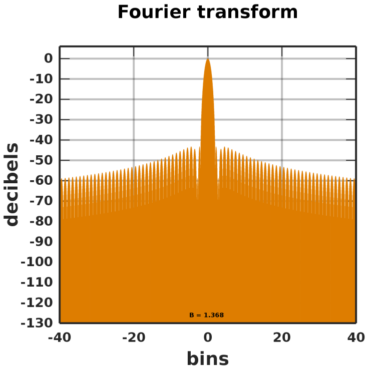
Hamming window spectrum

The frequency response of this filter is just the window DTFT, $W(\omega)$, shifted up to ω_k:

$$H_k(\omega) = W\left(\omega - \omega_k\right)$$

For a Hamming window, $w[n]$ is on the left, $W(\omega)$ is on the right:

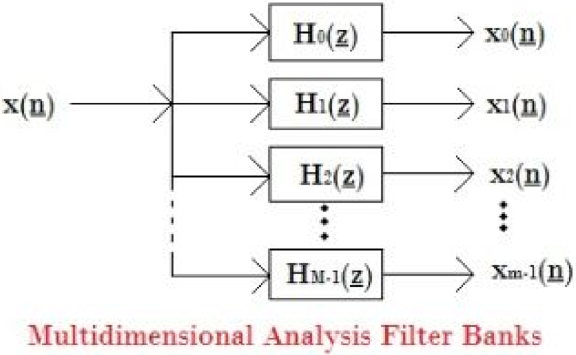




By Olli Niemitalo, public domain image,

STFT as a bank of analysis filters

So the STFT is just like filtering $x[n]$ through a bank of analysis filters, in which the k^{th} filter is a bandpass filter centered at ω_k:



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https://en.wikipedia.org/wiki/File:Multidimensional_Analysis_Filter_Banks.jpg

Short-Time Fourier Transform

- **STFT as a Transform:**

$$X[k,m] = \mathrm{DFT}\left\{w[n]x[n+m]\right\}$$

- **STFT as a Filterbank:**

$$X[k,m] = x[m] * h_k[-m], \qquad h_k[m] = w[m]e^{j\omega_k m}$$

Outline

Short-Time Fourier Transform

- **STFT as a Transform:**

$$X[k,m] = \text{DFT}\left\{w[n]x[n+m]\right\}$$

- **STFT as a Filterbank:**

$$X[k,m] = x[m] * h_k[-m], \qquad h_k[m] = w[m]e^{j\omega_k m}$$

The inverse STFT

STFT as a transform is defined as:

$$X[k,m] = \sum_{n=m}^{m+(N-1)} w[n-m]x[n]e^{-j2\pi k(n-m)/N}$$

Obviously, we can inverse transform as:

$$x[n] = \frac{1}{Nw[n-m]} \sum_{k=0}^{N-1} X[k,m]e^{j2\pi k(n-m)/N}$$

The inverse STFT

We get a better estimate of $x[n]$ if we average over all of the windows for which $w[n-m] \neq 0$. Remember that this happens when $0 \leq n-m \leq (N-1)$, so

$$x[n] = \frac{\sum_{m=n-(N-1)}^{n} \frac{1}{N} \sum_{k=0}^{N-1} X[k,m] e^{j\omega_k (n-m)}}{\sum_{m=n-(N-1)}^{n} w[n-m]}$$

The denominator is

$$W(0) = \sum_{m=0}^{N-1} w[m]$$

So

$$x[n] = \frac{1}{NW(0)} \sum_{m=n-(N-1)}^{n} \sum_{k=0}^{N-1} X[k,m] e^{j\omega_k (n-m)}$$

STFT: Forward and Inverse

- **Short Time Fourier Transform (STFT)**:

$$X[k,m] = \sum_{n=m}^{m+(N-1)} w[n-m]x[n]e^{-j\omega_k(n-m)}, \quad \omega_k = \frac{2\pi k}{N}$$

- **Inverse Short Time Fourier Transform (ISTFT)**:

$$x[n] = \frac{1}{NW(0)} \sum_{m=n-(N-1)}^{n} \sum_{k=0}^{N-1} X[k,m]e^{j\omega_k(n-m)}$$

ISTFT as a bank of synthesis filters

Inverse Short Time Fourier Transform (ISTFT):

$$x[n] = \frac{1}{NW(0)} \sum_{m=n-(N-1)}^{n} \sum_{k=0}^{N-1} X[k,m] e^{j\omega_k(n-m)}$$

The ISTFT is the sum of filters:

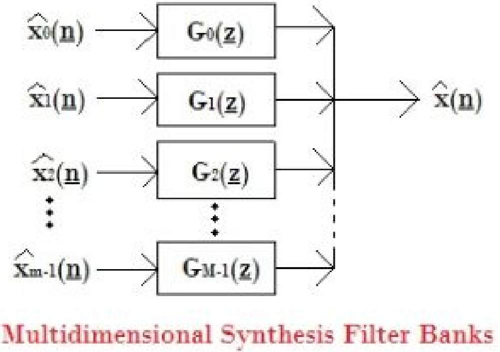
$$x[n] = \frac{1}{W(0)} \sum_{m=n-(N-1)}^{n} \sum_{k=0}^{N-1} X[k,m] e^{j\omega_k(n-m)}$$
$$= \sum_{k=0}^{N-1} \left(X[k,m] * g_k[m] \right)$$

where

$$g_k[m] = \begin{cases} \frac{1}{W(0)} e^{j\omega_k m} & 0 \le m \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

ISTFT as a bank of synthesis filters

So the ISTFT is just like filtering $X[k, m]$ through a bank of synthesis filters, in which the k^{th} filter is a bandpass filter centered at ω_k:



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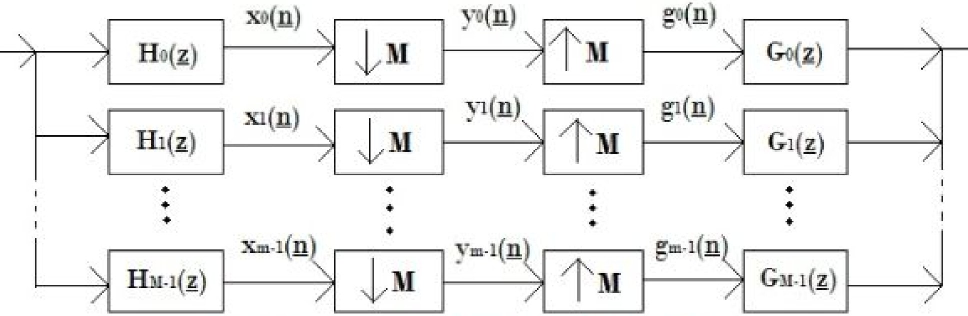
https://en.wikipedia.org/wiki/File:Multidimensional_Synthesis_Filter_Banks.jpg

The whole process: STFT and ISTFT as a filterbanks

We can compute the STFT, downsample, do stuff to it, upsample, and then resynthesize the resulting waveform:



Multidimensional M_Channel Filter Banks

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https://en.wikipedia.org/wiki/File:Multidimensional_M_Channel_Filter_Banks.jpg

Outline

Short-Time Fourier Transform

- **STFT as a Transform:**

$$X[k, m] = \text{DFT}\left\{w[n]x[n+m]\right\}$$

- **STFT as a Filterbank:**

$$X[k, m] = x[m] * h_k[-m], \quad h_k[m] = w[m]e^{j\omega_k m}$$

Relative Benefits of Transforms vs. Filters

- **STFT as a Transform:** Implement using Fast Fourier Transform.

$$X[k,m] = \text{DFT}\left\{w[n]x[n+m]\right\}$$
$$\textbf{Computational Complexity} = \mathcal{O}\left\{N\log_2(N)\right\}\ \text{per}\ m$$
$$\textbf{Example:} N = 1024$$
$$\textbf{Computational Complexity} = 10240\ \text{multiplies/sample}$$

- **STFT as a Filterbank:** Implement using convolution.

$$X[k,m] = x[m] * h_k[-m]$$
$$\textbf{Computational Complexity} = \mathcal{O}\left\{N^2\right\}\ \text{per}\ m$$
$$\textbf{Example:} N = 1024$$
$$\textbf{Computational Complexity} = 1048576\ \text{multiplies/sample}$$

What about other filters?

- Obviously, FFT is much faster than the convolution approach.
- Can we use the FFT to speed up other types of filter computations, as well?
- For example, how about gammatone filters? Could we compute those from the STFT?

What about other filters?

- We want to find $y[n] = f[n] * x[n]$, where $f[n]$ is a length-N impulse response.
- Complexity of the convolution in time domain is $\mathcal{O}\{N\}$ per output sample.
- We can't find $y[n]$ exactly, but we can find $\tilde{y}[n] = f[n] \circledast (w[n-m]x[n])$ from the STFT:

$$Y[k,m] = F[k]X[k,m]$$

- It makes sense to do this only if $F[k]$ has far fewer than N non-zero terms (narrowband filter).

Bandpass-Filtered Signal Power

In particular, suppose that $f[n]$ is a bandpass filter, and we'd like to know how much power gets through it.

So we'd like to know the power of the signal $\tilde{y}[n] = f[n] \circledast (w[n-m]x[n])$. We can get that as

$$\begin{aligned}\sum_{n=0}^{N-1} \tilde{y}[n]^2 &= \frac{1}{N}\sum_{k=0}^{N-1} |Y[k,m]|^2 \\ &= \frac{1}{N}\sum_{k=0}^{N-1} |F[k]|^2 |X[k,m]|^2\end{aligned}$$

Outline

Summary

- **STFT as a Transform:**

$$X[k,m] = \sum_{n=m}^{m+(N-1)} w[n-m]x[n]e^{-j\omega_k(n-m)}, \quad \omega_k = \frac{2\pi k}{N}$$

- **STFT as a Filterbank:**

$$X[k,m] = x[m] * h_k[-m], \qquad h_k[m] = w[m]e^{j\omega_k m}$$

- **Other filters using STFT:**

$$\text{DFT}\left\{f[n] \circledast (w[n-m]x[n])\right\} = H[k]X[k,m]$$

- **Bandpass-Filtered Signal Power**

$$\sum_{n=0}^{N-1} \tilde{y}[n]^2 = \frac{1}{N}\sum_{k=0}^{N-1} |F[k]|^2 |X[k,m]|^2$$