TOLERANCE ANALYSIS

Introduction

There is a finite precision in the components which comprise your project. Manufacturers specify these limitations in their datasheets. The manufacturing process may limit the accuracy of the components. For example, capacitors produced in a single run may have a capacitance which varies up to 20% from the rated value. The performance of sensors and actuators is limited by either physical or technical constraints. An ultrasonic sensor may only give a reading accurate to within a few millimeters or a servo may only be able to point to within 1 degree of its commanded position. In the tolerance analysis, you will show that your project will still be able to function despite the limited accuracy of the sensors, actuators, component values, and so forth.

Tolerance analysis may also be viewed from the perspective of manufacturing. If you have designed your project correctly, it should function with the specified nominal component values. One could imagine that small shifts of the component values, such as substituting a 330.002 Ω resistor in place of a 330.000 Ω one, should not affect the operation of most circuits. How much can the component values be varied before the performance of the device is compromised? This is not an idle question. You can save several cents by buying a resistor rated at 10% tolerance instead of one rated at 0.1% tolerance . While the cost difference is insignificant for a single prototype, if you are manufacturing millions of devices, each with dozens of components, a difference of a few cents per component amounts to a significant sum of money. So ask yourselves what precision you need for your component values, how many bits you need from your A/D converter, and how accurate your sensor or actuator need to be.

Although it may be presented as the analysis of a particular implementation, the tolerance analysis is actually part of the design process. A good design method is outlined here:

1. Create a simulation with your component values as parameters.
2. Select components and find or estimate their tolerances.
3. Run the simulation over the range of parameters specified by your parts' tolerances.
4. Iterate this procedure, adjusting the tolerances specified, until your requirement(s) is(are) satisfied over the entire range.
5. If it is not possible or practical to further reduce the tolerance of parts, it will be necessary to alter your design.

After you have finished your tolerance analysis, you should know precisely what is limiting your design's performance. If someone asked you to improve the performance, you will know what to change.

As it constitutes the proof that your design will succeed, the tolerance analysis is one of the most import sections of your design review document. You should perform a tolerance analysis on a component of your project which is critical to the core functionality of your project. Your tolerance analysis should contain:

1. A statement identifying the most critical overall feature(s) of your project.
2. The components (which may span several blocks of your design) involved in the overall requirement and their accuracy, either specified by the manufacturer or empirically measured (or estimated if you don't have them).
3. A discussion of the insights you developed by performing the analysis.

Feel free to add plots and tables to help you explain your analysis.

Below are two examples of tolerance analyses. The short project description preceding each example is only to provide context and is not necessary in your tolerance analysis section. In your paper, the project description will be in the introduction to the paper.

Example #1: Anti-Aliasing RC Filter

Large construction projects frequently generate ground vibrations, which result from moving heavy equipment and driving piles. Neighbors complain about lack of sleep, broken windows

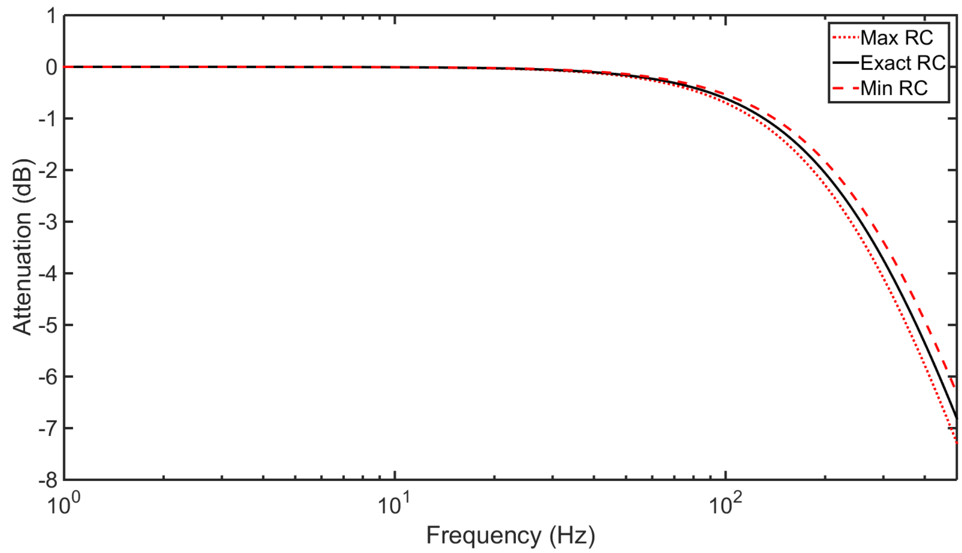

Figure 1: Plot of the transfer function $T(f)$ for the exact and extremal values of the RC product.

and cracked foundations. Our project is a tool for construction companies to monitor the level of vibrations which propagate off the construction site. It will enable them to mitigate the vibrations and avoid costly lawsuits. The devices will produce calibrated plots of the intensity of the vibrations as a function of frequency from 0.1-150 Hz, the range over which our research indicates waves will propagate. They will be battery powered and transmit the vibration data to a base station every hour.

A critical component of this project is the anti-aliasing filter before the A/D conversion. There are three requirements for the filter: 1) It must have an attenuation of at least 6 dB for frequencies at or above the Nyquist frequency. 2) As a calibrated measurement is required, unit-to-unit changes in the transfer function due to variations in the analog components must be less than 0.43 dB (5% intensity). 3) The flatness of the passband from 0-150 Hz must be less than 3 dB.

To achieve the three requirements listed in the previous paragraph, a 10 kΩ resistor with 1% tolerance and a 62 nF capacitor with 6% tolerance were chosen. The sampling rate of the A/D converter is 1000 Hz. It will now be demonstrated that this design will meet the requirements even with the component tolerances. The transfer function for an RC lowpass filter is given by

$$T(f) = 20\log_{10}\left|\frac{1}{1 + j2\pi fRC}\right|. \tag{1}$$

It is apparent from Equation 1 that only the product RC and not the individual values R and C is important in determining the shape of the transfer function. Figure 1 plots the transfer functions with the maximum, exact, and minimum RC products. It is apparent from looking at the plots that both the lowest value of the transfer function and the greatest variation among the transfer functions in the passband occur at 150 Hz.

Table 1 summarizes several data points from Figure 1. By examining the 500 Hz column, it is apparent that requirement 1 will be fulfilled in all cases. From the 150 Hz (where the largest variation in the pass band of the transfer functions occurs) column, it can be seen that the variation in the passband is from -1.43 dB to -1.13 dB, a change of 0.30 dB (3.6 % intensity). Therefore, requirement 2 is fulfilled. Finally, it is obvious from the 0 Hz and 150 Hz columns of Table 1 that requirement 3 will be fulfilled.

Table 1: Summary of $T(f)$ at several frequencies for the exact and extremal values of the RC product.

	0 Hz	150 Hz	500 Hz
Max RC	0 dB	-1.43 dB	-7.28 dB
Exact RC	0 dB	-1.28 dB	-6.81 dB
Min RC	0 dB	-1.13 Db	-6.32 dB

It is possible to find other combinations of resistor and capacitor tolerances which will still meet the specifications, for example a 20% tolerance resistor and a 1% tolerance capacitor. This option was not chosen because precision resistors are less expensive than precision capacitors. During the course of the analysis, it was discovered that the sampling rate needed to be increased from 500 Hz to 1000 Hz to meet requirements 1 and 2 simultaneously.. While the sampling rate could have been kept at 500 Hz using a higher-order filter, increasing the sampling rate allowed the design to function without adding additional components. High frequency noise which could be aliased to lower frequencies could be attenuated even further by again increasing the sampling rate, but this would increase the burden on the processor.

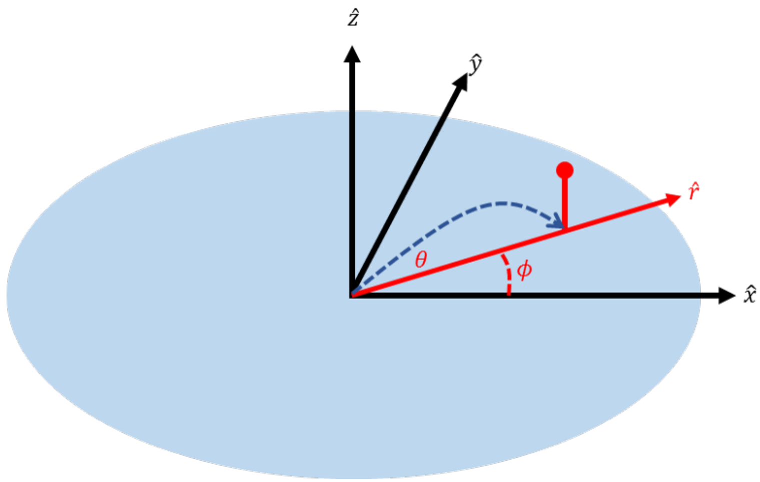

Figure 2: Physical layout of the catapult system. The catapult is at the origin and the target is marked with the red pole. The line from the origin to the base of the target defines the $\hat{r}$ axis. The azimuth ϕ is the angle between the $\hat{r}$ and $\hat{x}$ axes. Note the elevation, or firing angle, θ is measured from the $\hat{r}$ and not the $\hat{z}$ axis. A ballistic trajectory is shown in blue.

Example #2: Miniature Medieval Weapon

Inspired by our love for medieval weaponry, we are creating a miniature catapult which fires $\frac{1}{2}$" steel ball bearings. Sensors find the range and azimuth of the target. The sensor data is processed by a microcontroller which then aims the catapult. A servo controls the azimuth while a stepper motor controls the elevation. After firing, the catapult will automatically reload and recompress the spring.

The most critical aspect of our project's performance is that it is able to hit its target. Our requirement is that the bearing strike within 10 cm of any target within the 5 m range of the catapult. For the tolerance analysis, we will determine the total worst-case error in the impact position of the ball bearing. This analysis will comprise two sections: First, the worst case error along the azimuthal ($\hat{\phi}$) direction will be examined. Second, the worst case error of the range (along the $\hat{r}$ direction) will be calculated. For reference, the physical layout of the coordinate system is shown in Figure 2.

Azimuth Error

The catapult uses a laser sight aligned with the catapult to determine the azimuth of a target. The entire catapult is rotated until the back reflection signal from the target is maximized. The accuracy is limited by the step size of the azimuth servo, which is 1°. The worst case azimuth error is one half the step size, or 0.5°. As a function of range r, the worst case position error along the azimuthal direction is

$$\Delta a = r\Delta\phi, \tag{2}$$

where $\Delta\phi = \frac{\pi}{360}$ rad. Equation 2 is justified because $\Delta\phi$ is small.

Range Error

For our range sensor, we are using the analog output pin from an ultrasonic sensor (HRLV MaxSonar EZ4). The manufacturer specifies that the unit is accurate to 1 mm and its distance-to-voltage calibration is α=4.88 mV/5 mm. The analog voltage will be read with one of the 10-bit analog inputs to our ATMega328p microcontroller. The A/D converter of the microcontroller has a 5 V range, which means the digitization step size is

$$V_{step} = \frac{5\text{ V}}{2^{10}\text{ steps}} = 4.88\text{ mV}. \tag{3}$$

In the worst case, the digitization error is equal to one half of the voltage step size. The corresponding error in the measured distance is:

$$\Delta r_{AD} = \frac{V_{step}}{2\alpha} = 2.5\text{ mm}. \tag{4}$$

Given the limited accuracy of the sensor itself and the digitization noise, the worst case error in finding the range to the target is

$$\Delta r_{sensor/AD} = \pm 3.5\text{ mm}. \tag{5}$$

Now we will examine the physics of firing the catapult. From elementary physics, we know the trajectory in the $\hat{z}$-direction, given an initial upwards velocity v_z from an initial height of zero is

$$z(t) = -\frac{1}{2}gt^2 + v_z t. \tag{6}$$

Solving for the time of flight, we find that

$$t_{impact} = \frac{2v_z}{g}. \tag{7}$$

The corresponding distance the projectile travels down range is

$$r_{impact} = v_r t_{impact} = \frac{2v_r v_z}{g} = \frac{v^2 \sin 2\theta}{g}, \tag{8}$$

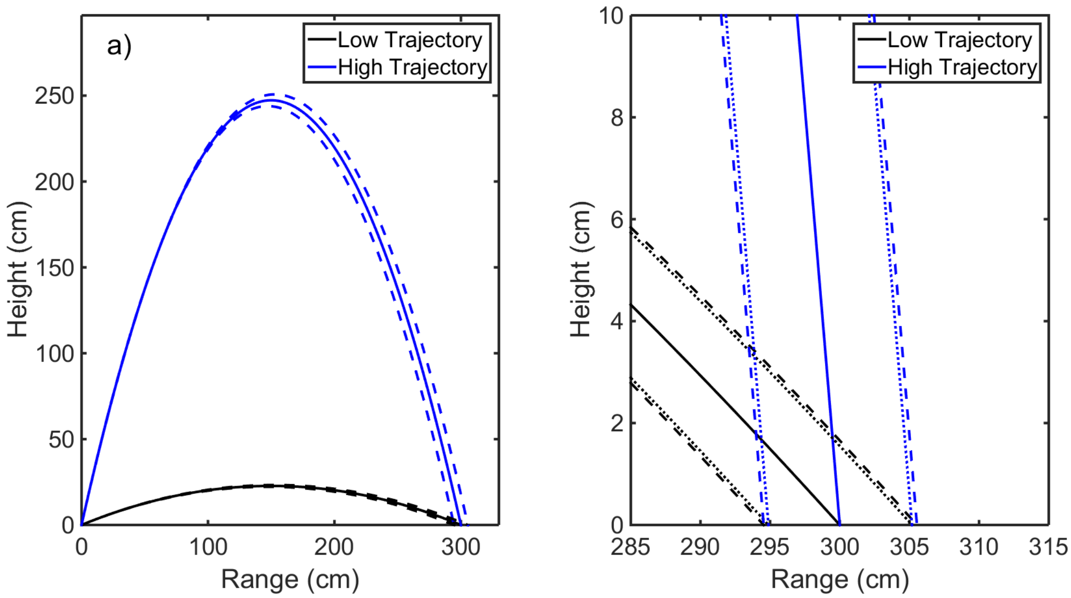

Figure 3: (a) Graph plotting the two firing trajectories which strike the same target located 300 cm from the catapult. The dashed lines show the worst-case error in the trajectory. (b) Closeup of the trajectories near the impact point. The dotted lines show the error which results from only the firing mechanism, while the dashed lines show the total error. Note the error is equal for both trajectories.

where we have used $v_r = v\cos\theta$ and $v_z = v\sin\theta$ to create the simplified form of the equation on the right. Using the relationship

$$E = \frac{1}{2}mv^2, \tag{9}$$

which is true as the bearing leaves the catapult, an equation describing the impact position of the bearing in terms of the its mass m and the initial energy E may be found. It is

$$r_{impact} = \frac{2E}{mg}\sin 2\theta. \tag{10}$$

Therefore, to strike a target at r_{target}, with a bearing mass m fired with an initial energy E, the firing angle should be adjusted to

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{mgr_{target}}{2E}\right). \tag{11}$$

Looking at Equation 10 more closely, we notice there is actually a second firing angle which will strike the same target: $\theta' = \pi/2 - \theta$. Plots of both trajectories are shown in Figure 3a.

Various uncertainties present in the catapult mechanism and projectile result in discrepancies between the desired and actual impact locations. During test firing of the catapult, we found that it shot straight, but there was an uncertainty in the exit energy of ± 2.3 mJ when the catapult was tuned to shoot at 249 mJ. There may also be some variation in the mass of the bearings, which was not specified by the manufacturer. Using a digital scale, we measured the mass of all 100 of our bearings to be 8.4 g. Due to the limited accuracy of the scale, variations in mass among bearings as large as $\pm$ 0.05 g will go undetected and may be present. A final source of uncertainty is the limited pointing precision of the stepper motor. Our stepper motor has 3600 steps per rotation, so the expected 'quantization' pointing error will be $\pm 0.05°$.

The impacts of these various uncertainties can be analyzed by taking the total derivative of Equation 10, which is

$$\Delta r_{firing} = \frac{2}{mg}\Delta E\sin 2\theta - \frac{2E}{m^2 g}\Delta m\sin 2\theta + \frac{4E}{mg}\Delta\theta\cos 2\theta. \tag{12}$$

The total range error is the sum of the error from measuring the target distance and the error in the firing process

$$\Delta r = \Delta r_{sensor/AD} + \Delta r_{firing}. \tag{13}$$

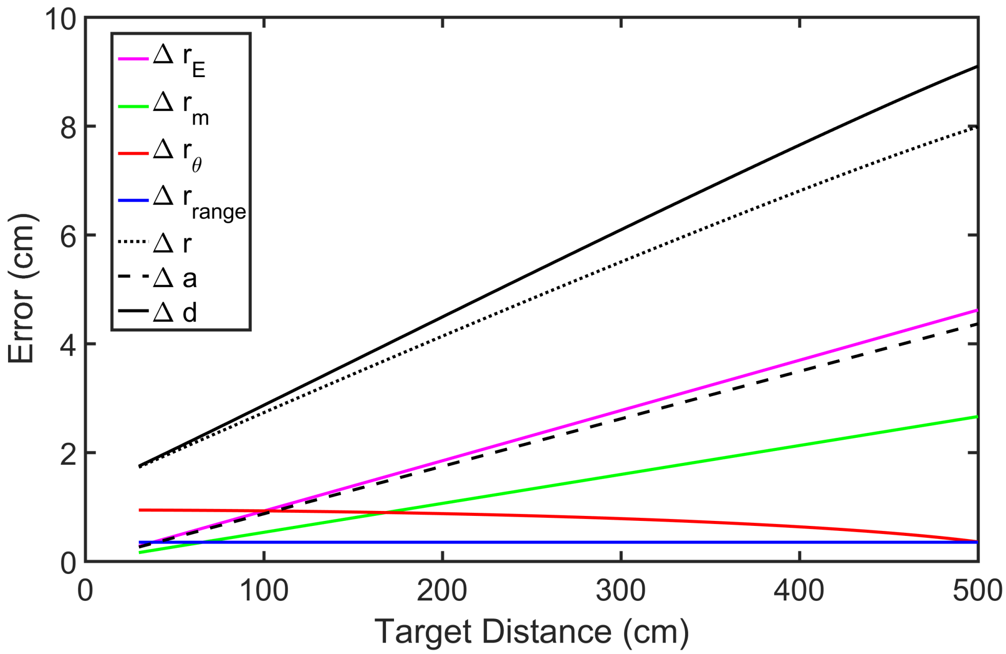

Figure 4: Worst case errors from various sources plotted as a function of target range. The quantities Δr_E, Δr_m, and Δr_θ are the individual terms of Equation 12. From Equation 5, the range error Δr_{range} is plotted. The total range error Δr, total azimuth error Δa, and total error Δd are plotted from Equations 13, 2, and 14, respectively.

Plots of the actual trajectories along $\hat{r}$ in the worst case are show in Figure 3 a and b. Note the error is equal for both the upper and lower trajectories.

Total Error

Because the error in the azimuthal direction Δa and the error in the range Δr are perpendicular to each other, the total distance Δd by which the bearing misses the target is given by

$$\Delta d = \sqrt{\Delta a^2 + \Delta r^2}. \tag{14}$$

Figure 4 shows how uncertainties in various parameters combine to generate the total distance by which the bearing misses the target. All quantities are plotted in their worst case.

The maximum worst case error in the impact position is 9.01 cm which falls within the 10 cm specification. The average error is expected to be better. The components of the total error will be less than their worst case values and, in general, will not add together with the same sign. Finally, we were only able to set and upper bound for the variation of the masses of the bearings with our limited-precision scale. The actual variation the bearings' masses will likely be lower.

We would like to point out the tolerance analysis altered our design. Originally, we had specified a stepper motor with $\pm 1^\circ$ steps to control the firing angle, However, we found this caused an unacceptably large error in the range (± 18 cm) and replaced it with a more accurate stepper motor. If we wished to improve the accuracy of the project now, it would make the most sense to work on improving the spring mechanism, which is the largest component of the error in the current design.

Comments on Examples

The structures of tolerance analyses are quite diverse and project dependent. In the second example, all of the analysis was dedicated to showing the design could fulfill one requirement (accuracy), while the analysis of the first example showed the design could fulfill multiple requirements. In the first example, a single subsystem is examined. The second example spanned multiple subsystems (sensors, processing, actuators, and mechanical components). Both examples analyzed the worst case tolerance, which makes sense for the given projects. In your project, the average error or the distribution of the error may be more important.

You must use your judgment to determine what will be part of your tolerance analysis.