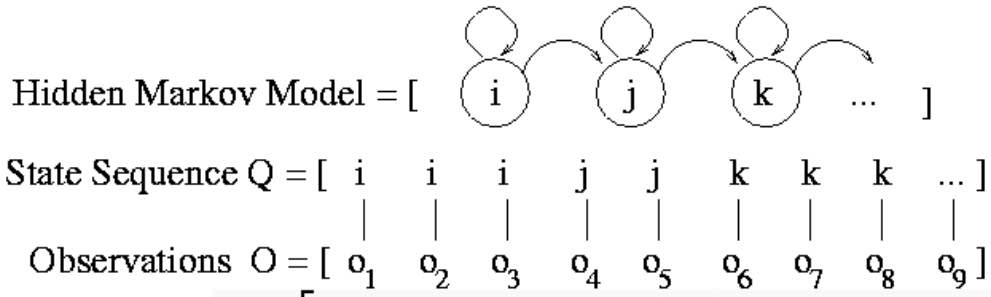
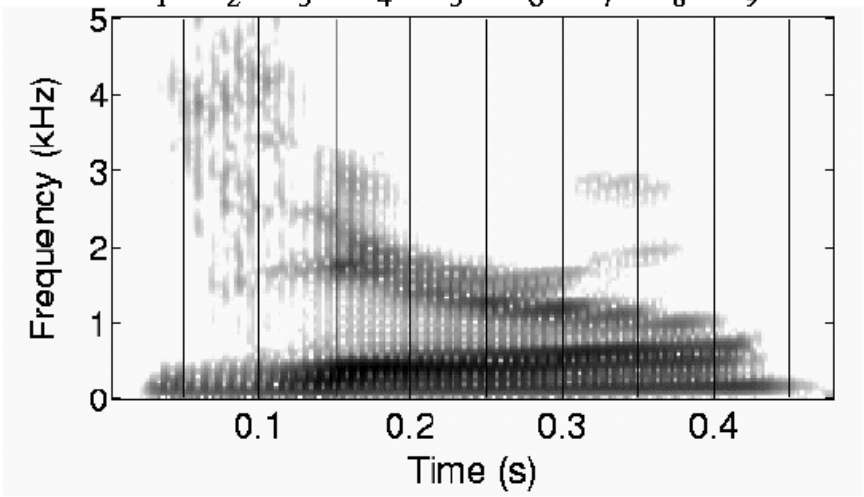
CS440/ECE448 Lecture 12: Hidden Markov Models

Mark Hasegawa-Johnson





Outline

- HMM: Probabilistic reasoning over time
- Viterbi algorithm

Review: Bayesian Classifier

- Class label $Y = y$, drawn from some set of labels
- Observation $X = x$, drawn from some set of features
- Bayesian classifier: choose the class label, y, that minimizes your probability of making a mistake:

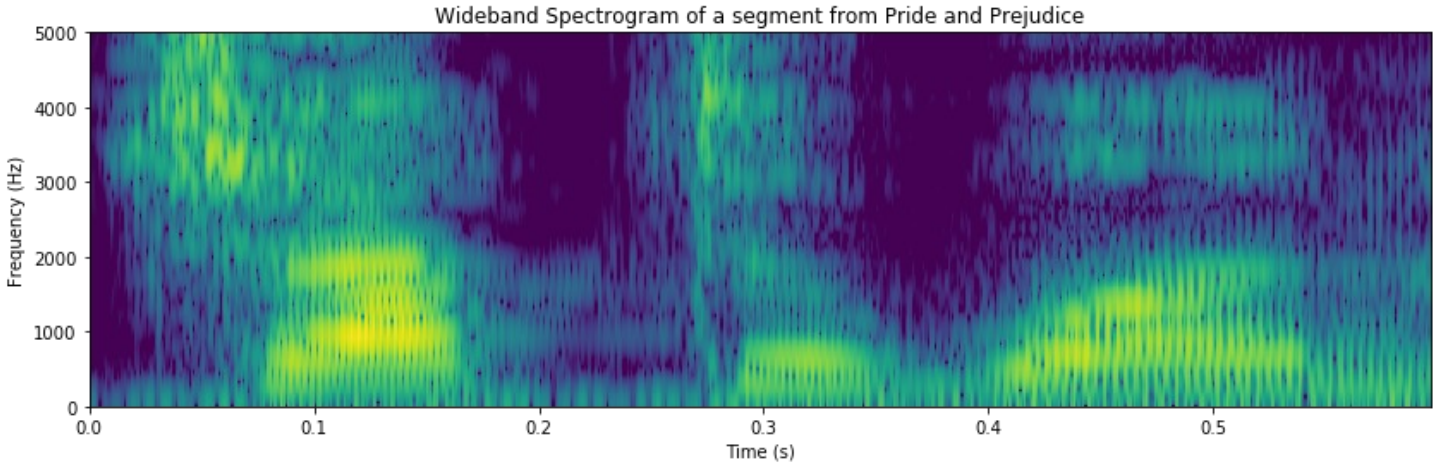
$$f(x) = \underset{y}{\operatorname{argmax}}\, P(Y = y | X = x)$$

Hidden Markov model: X and Y are sequences

- Class label sequence $Y = [Y_1, \dots, Y_T]$
- Observation sequence $X = [X_1, \dots, X_T]$
- Bayesian classifier: choose the class label sequence, $[y_1, \dots, y_T]$, that minimizes your probability of making a mistake:

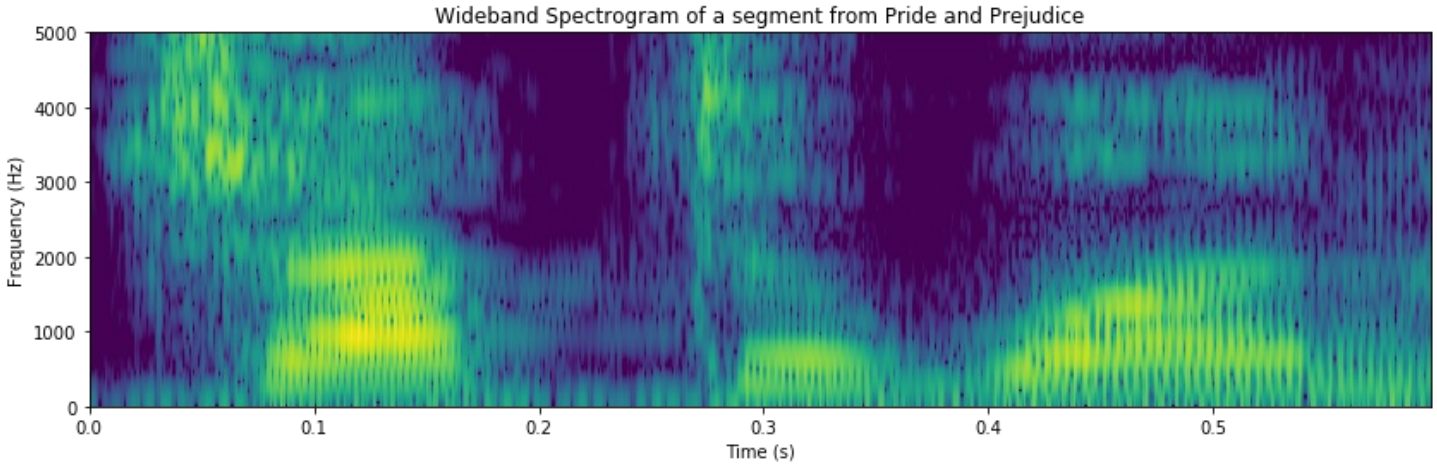
$$f(x) = \underset{y_1,\dots,y_T}{\operatorname{argmax}} P(Y = [y_1, \dots, y_T] | X = [x_1, \dots, x_T])$$

Example: Speech Recognition



- Here's a spectrogram of the utterance "chapter one."
- Each column is the Fourier transform of 0.02s of audio, spaced 0.01s apart. Let's call the spectral vector X_t, where t is time in centiseconds
- The speech sounds follow a sequence: silence for a while, then /sh/ for a while, then /ae/ for a while, then…. Let's denote the speech sound at time t as Y_t

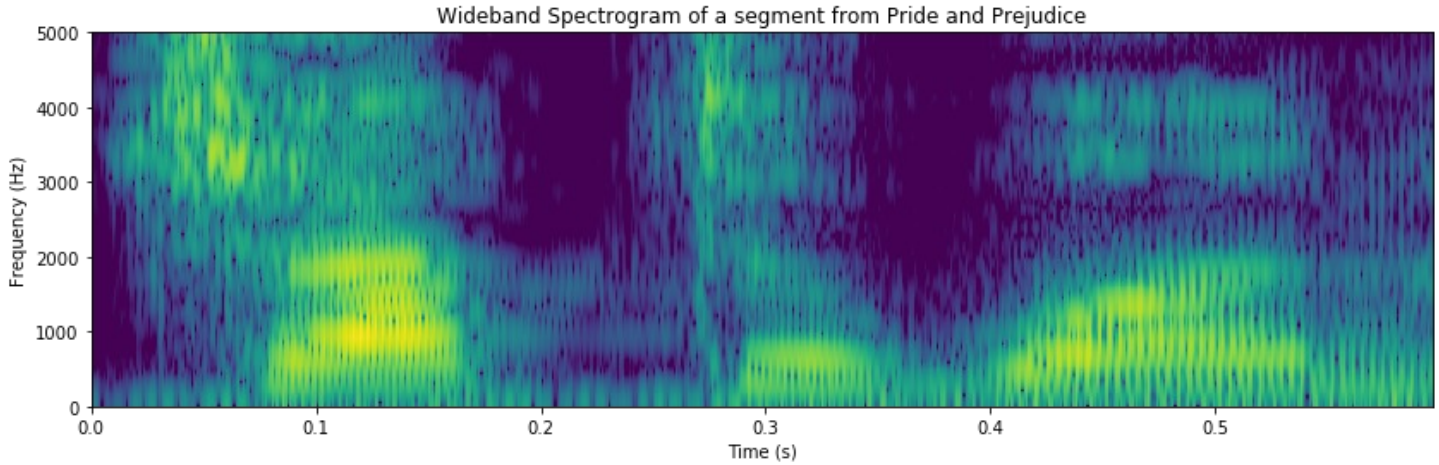
Hidden Markov Model

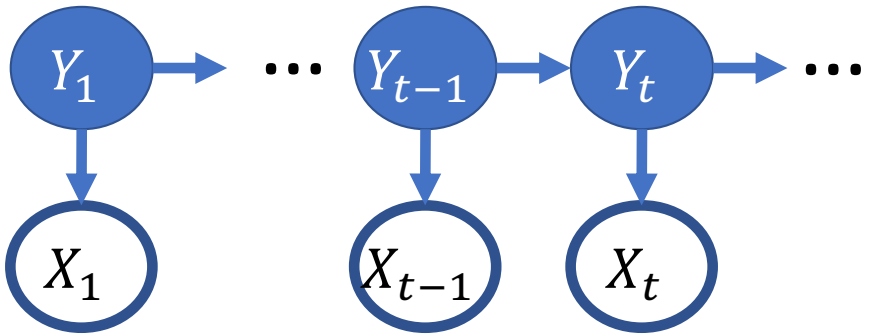


"Hidden Markov" Model:

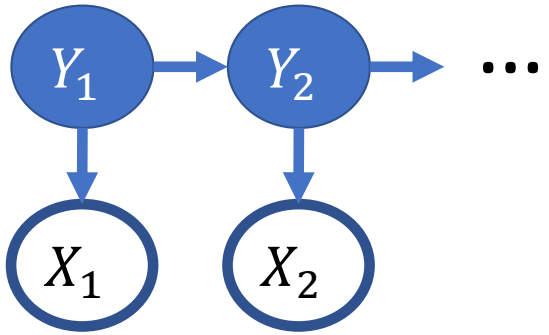
- **Hidden**: You don't know the label Y_t, instead, you only know the observation X_t, and the probabilities $P(X_t|Y_t)$
- **Markov**: Y_t depends only on Y_{t-1}, and you know $P(Y_t|Y_{t-1})$

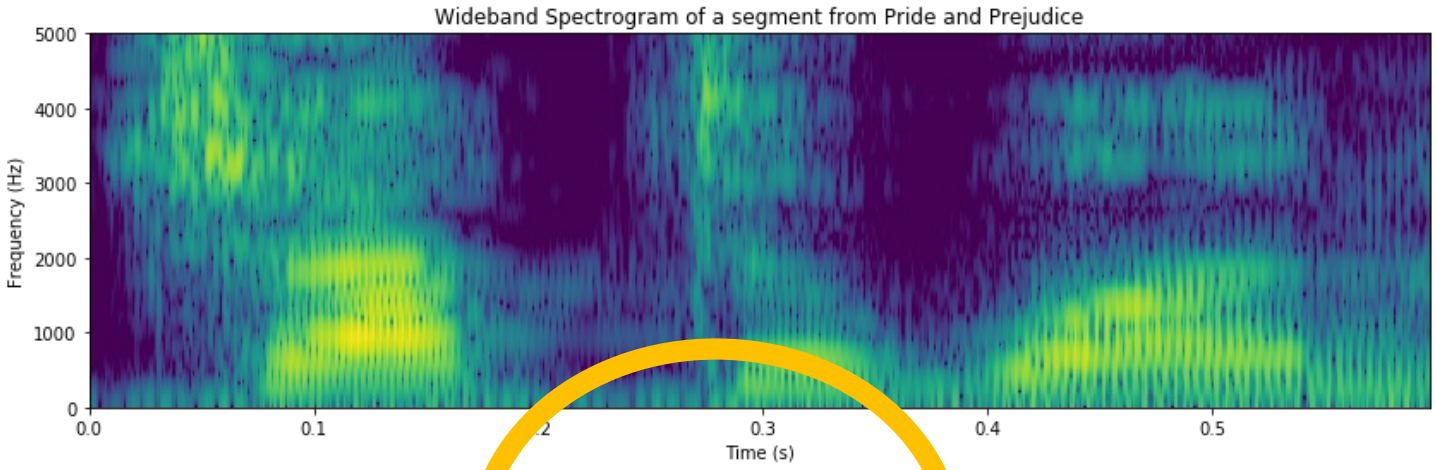
Hidden Markov Model





Hidden Markov Model





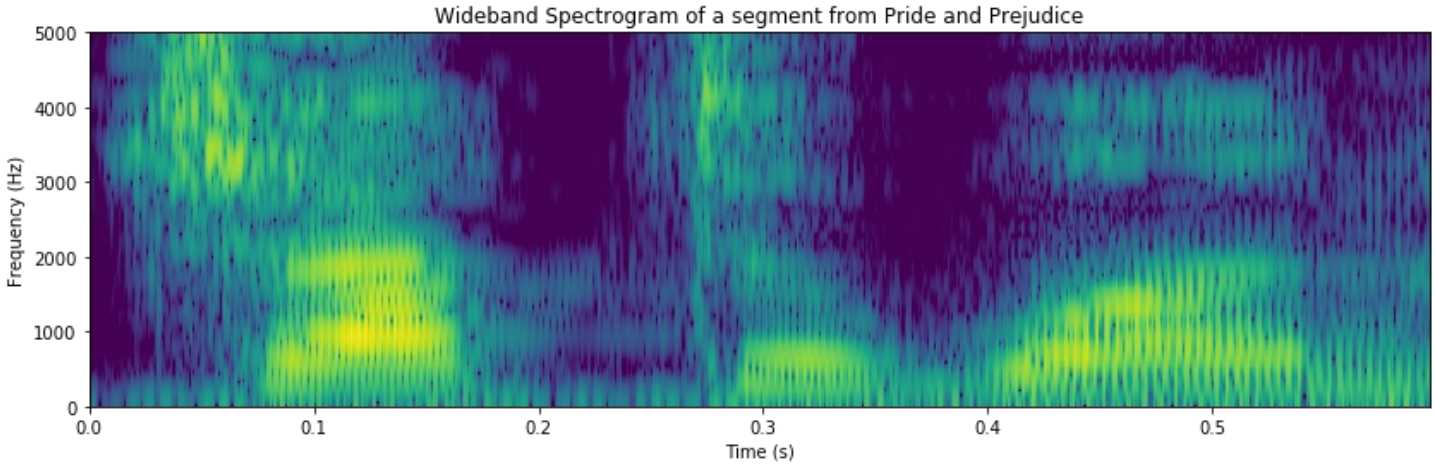
$$P(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d) = \sum_{y_1} \sum_{y_2} \cdots \sum_{y_d} P(y_1) P(\boldsymbol{x}_1|y_1) P(y_2|y_1) P(\boldsymbol{x}_2|y_2) P(y_3|y_2) \cdots$$

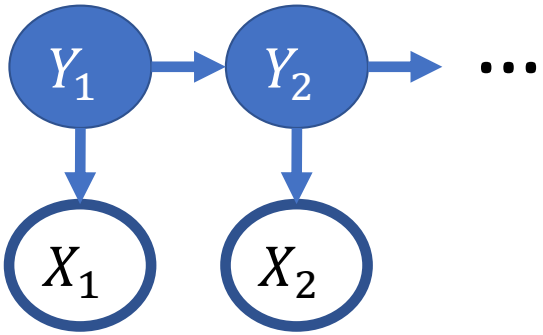
$\mathcal{O}\{|\mathcal{Y}|^d\}$ terms in this summation. Does this mean time-complexity is $\mathcal{O}\{|\mathcal{Y}|^d\}$?

The problem HMMs solve: Exponential complexity

- Suppose there are $|\mathcal{Y}|$ different speech sounds in English ($|\mathcal{Y}| \approx 50, d \approx 100$)
- The length of the utterance is d centiseconds ($d \approx 100$)
- Without the HMM assumptions, to compute $f(\boldsymbol{x}) = \text{argmax} P(y_1, \ldots, y_d | \boldsymbol{x}_1, \ldots, \boldsymbol{x}_d)$ requires a time complexity of $\mathcal{O}\{|\mathcal{Y}|^d\} \approx 50^{100}$

Hidden Markov Model



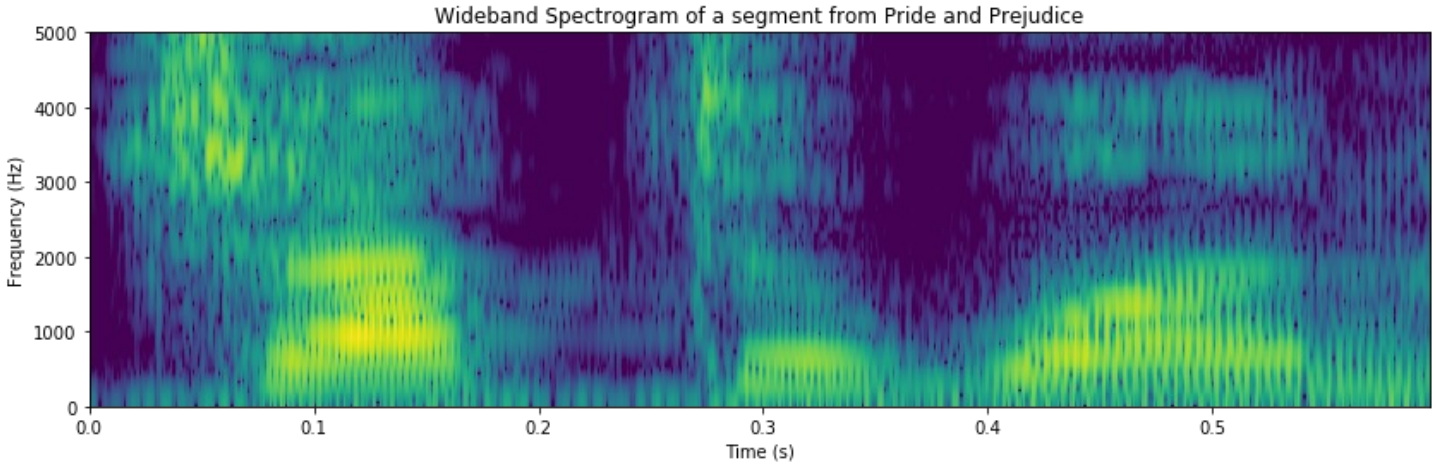


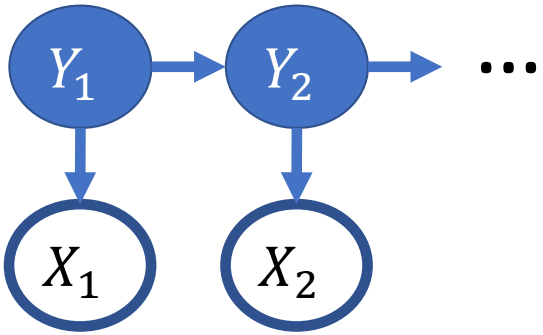
$$P(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d) = \sum_{y_1} \sum_{y_2} \cdots \sum_{y_d} P(y_1)P(\boldsymbol{x}_1|y_1)P(y_2|y_1)P(\boldsymbol{x}_2|y_2)P(y_3|y_2)\cdots$$

$$= \sum_{y_d} \cdots \sum_{y_2} P(y_3|y_2)P(\boldsymbol{x}_2|y_2) \sum_{y_1} P(y_2|y_1)P(\boldsymbol{x}_1|y_1)P(y_1)$$

$\mathcal{O}\{|\boldsymbol{Y}|\}$

Hidden Markov Model



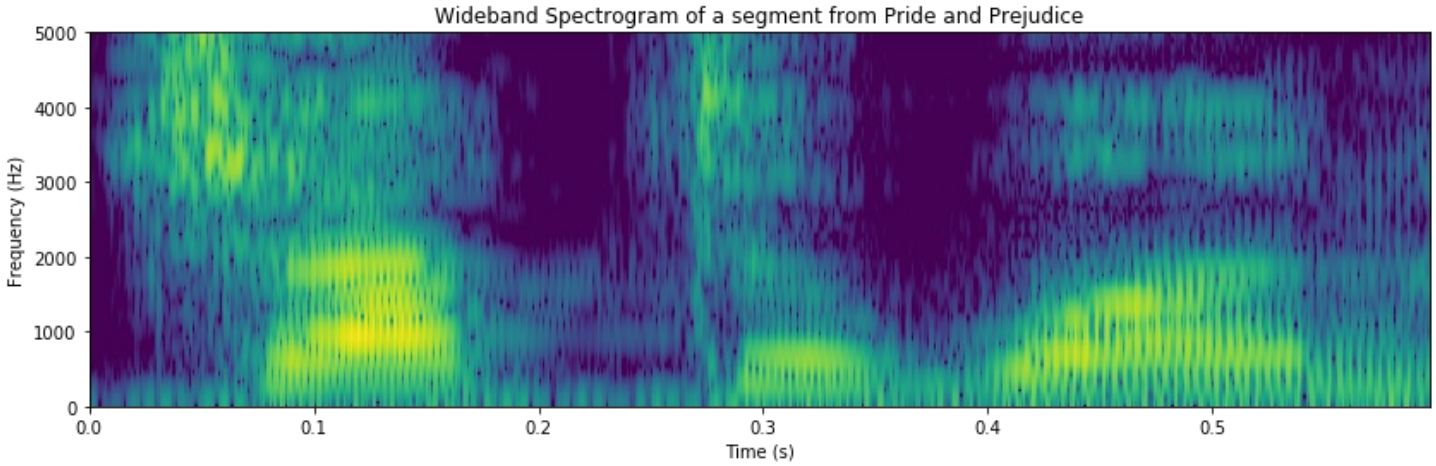


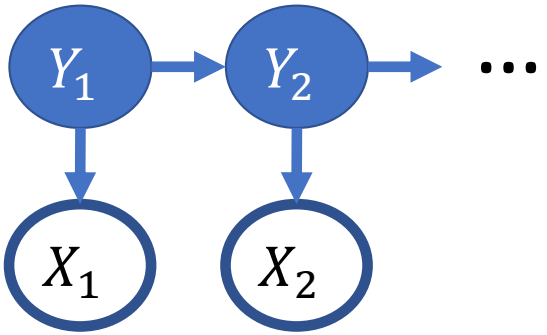
$$P(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d) = \sum_{y_1} \sum_{y_2} \cdots \sum_{y_d} P(y_1)P(\boldsymbol{x}_1|y_1)P(y_2|y_1)P(\boldsymbol{x}_2|y_2)P(y_3|y_2)\cdots$$

$$= \sum_{y_d} \cdots \sum_{y_2} P(y_3|y_2)P(\boldsymbol{x}_2|y_2) \sum_{y_1} P(y_2|y_1)P(\boldsymbol{x}_1|y_1)P(y_1)$$

$\boldsymbol{O\{|Y|^2\}}$

Hidden Markov Model





$$P(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d) = \sum_{y_1} \sum_{y_2} \cdots \sum_{y_d} P(y_1)P(\boldsymbol{x}_1|y_1)P(y_2|y_1)P(\boldsymbol{x}_2|y_2)P(y_3|y_2)\cdots$$

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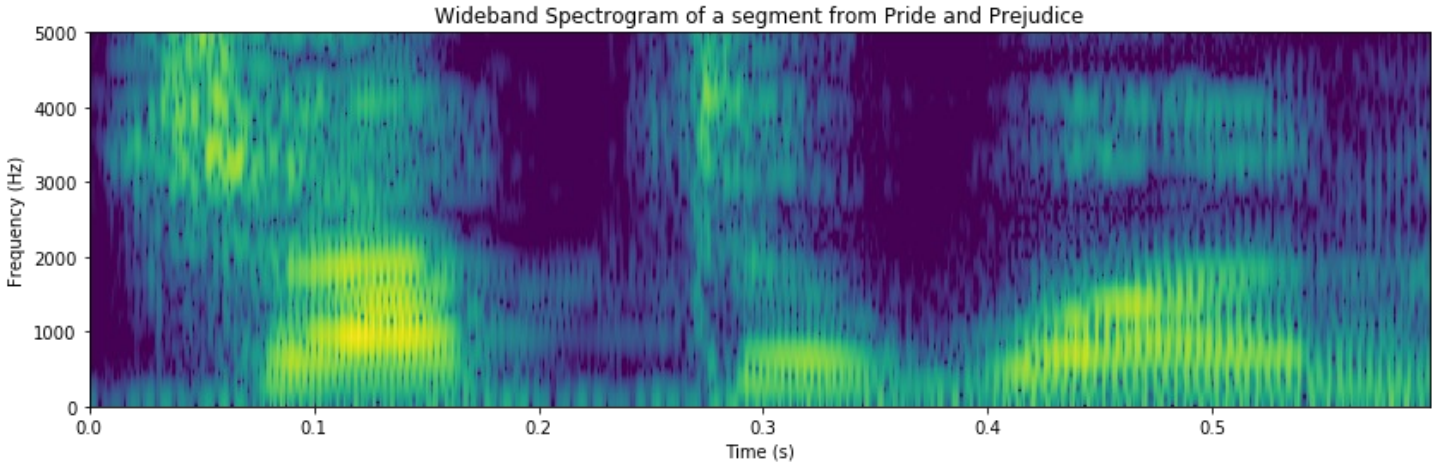
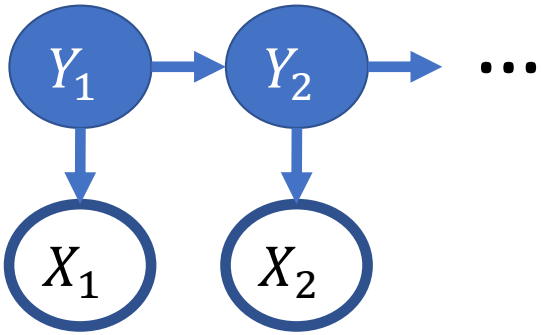
$$\mathcal{O}\{|\boldsymbol{Y}|\}$$

Hidden Markov Model

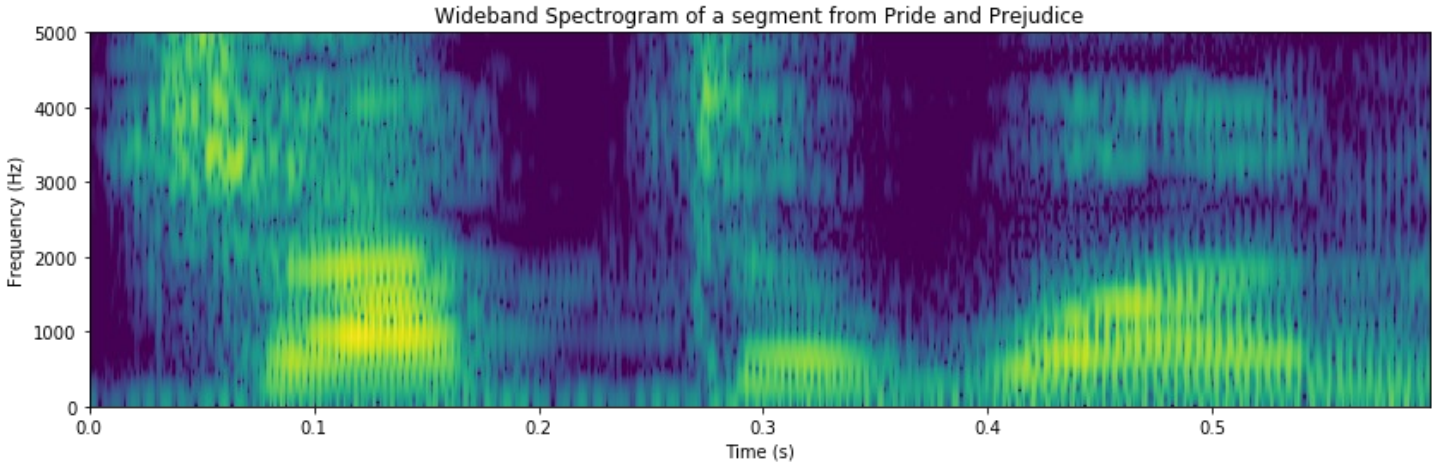


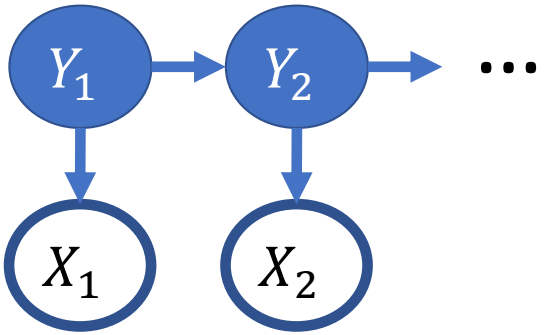


$$P(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d) = \sum_{y_1} \sum_{y_2} \cdots \sum_{y_d} P(y_1)P(\boldsymbol{x}_1|y_1)P(y_2|y_1)P(\boldsymbol{x}_2|y_2)P(y_3|y_2)\cdots$$

$$= \sum_{y_d} \cdots \sum_{y_2} P(y_3|y_2)P(\boldsymbol{x}_2|y_2) \sum_{y_1} P(y_2|y_1)P(\boldsymbol{x}_1|y_1)P(y_1) \quad \boldsymbol{O\{|y|^2\}}$$

Hidden Markov Model





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Key advantage of a hidden Markov model: Polynomial-time complexity

- A hidden Markov model makes the computation local, in the sense that each Y_t depends only on Y_{t-1}
- As a result, the computational complexity never gets larger than $\mathcal{O}\{|\mathcal{Y}|^2\}$

…and it works much better than naïve Bayes.

Claude Shannon (1948) gave these examples:

- Text generated by a naïve Bayes model (unigram model):

Representing and speedily is an good apt or come can different natural here he the a in came the to of to expert gray come to furnishes the line message had be these…

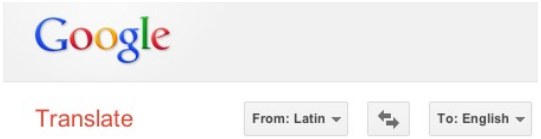
- Text generated by a HMM (bigram model):

The head and in frontal attack on an English writer that the character of this point is therefore another for the letters that the time of who ever told the problem for an unexpected…

Applications of HMMs

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)

- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are cross-lingual alignments



- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map

Source: Tamara Berg

Outline

- HMM: Probabilistic reasoning over time
- Viterbi algorithm

Viterbi Algorithm

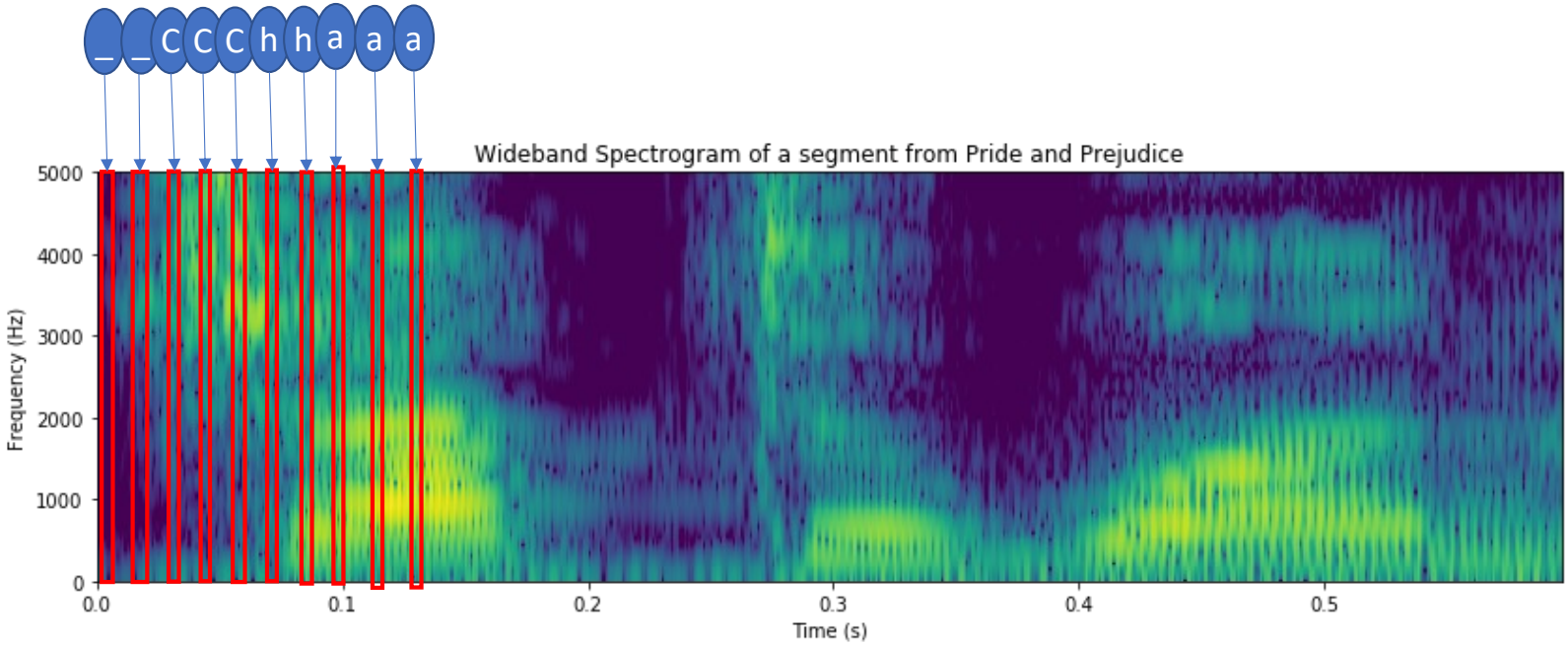
The Viterbi algorithm is a computationally efficient algorithm for computing the maximum *a posteriori* (MAP) state sequence,

$$f(\boldsymbol{x}) = \text{argmax}_{y_1,\ldots,y_d} P(y_1, \ldots, y_d | \boldsymbol{x}_1, \ldots, \boldsymbol{x}_d)$$

Example: Speech Recognition

- Observations: X_t = spectrum of 25ms frame of the speech signal.
- State: Y_t = phoneme or letter being currently produced

The goal of speech recognition: find the most probable sequence of text characters $\{y_1, \ldots, y_T\}$ given observations $\{X_1 = x_1, \ldots, X_T = x_T\}$.

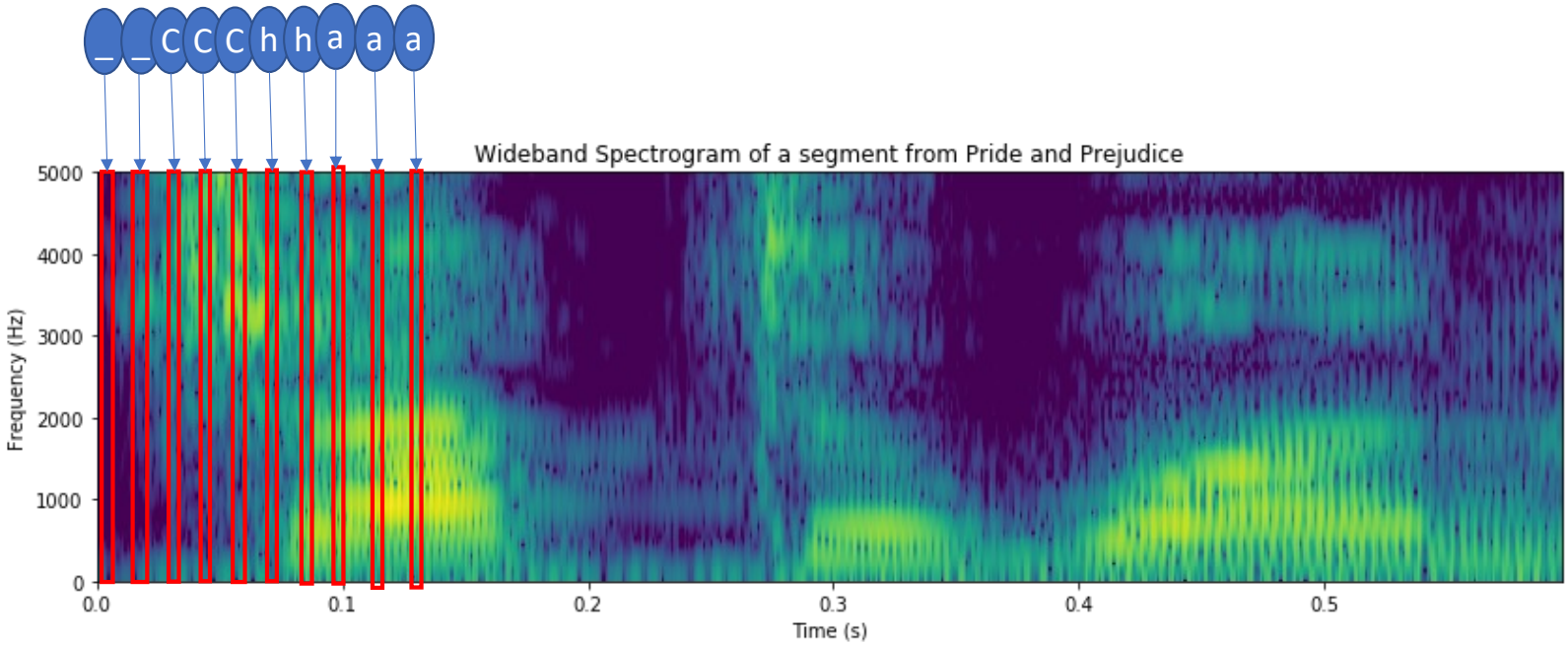


Viterbi Algorithm

Basic concept:

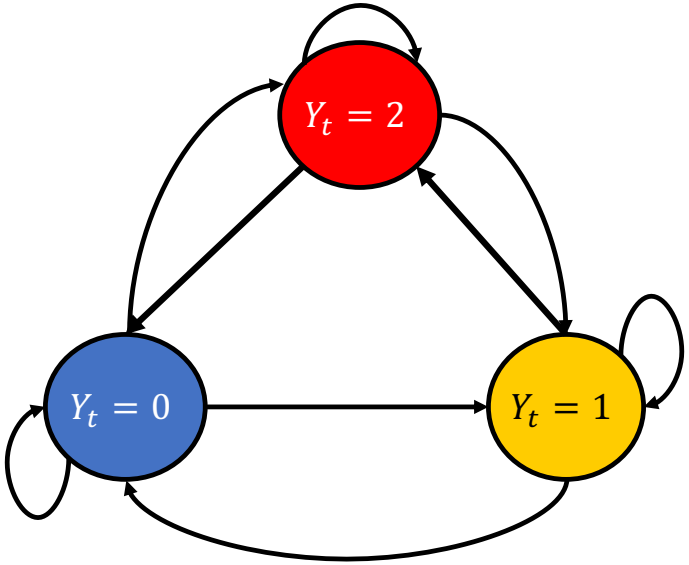
$$\max P(y_1, \ldots, y_d, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_d)$$

$$= \max_{y_d} \cdots \max_{y_2} P(y_3|y_2) P(\boldsymbol{x}_2|y_2) \max_{y_1} P(y_2|y_1) P(\boldsymbol{x}_1|y_1) P(y_1)$$



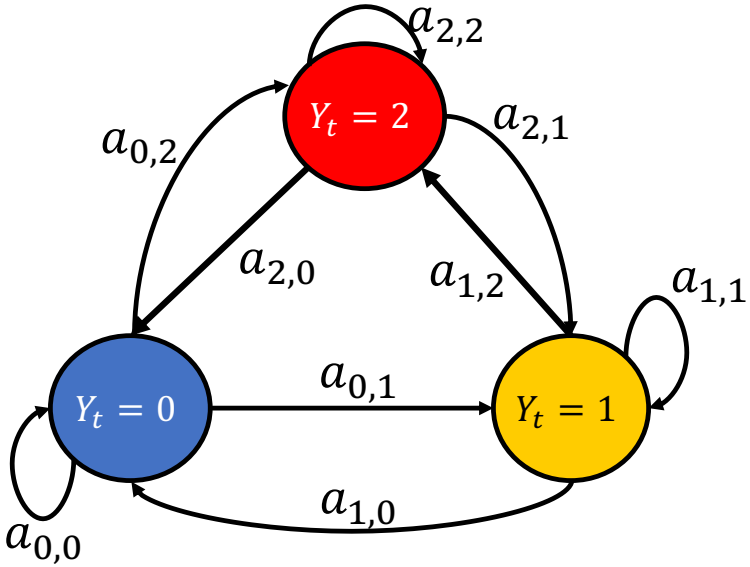
Finite State Model

- "State" = one of the values that Y_t can take
- "Edge" = a possible transition
- The time-complexity of inference is $\mathcal{O}\{|\mathcal{Y}|^2\}$ (because that's the number of edges in the FSM diagram)



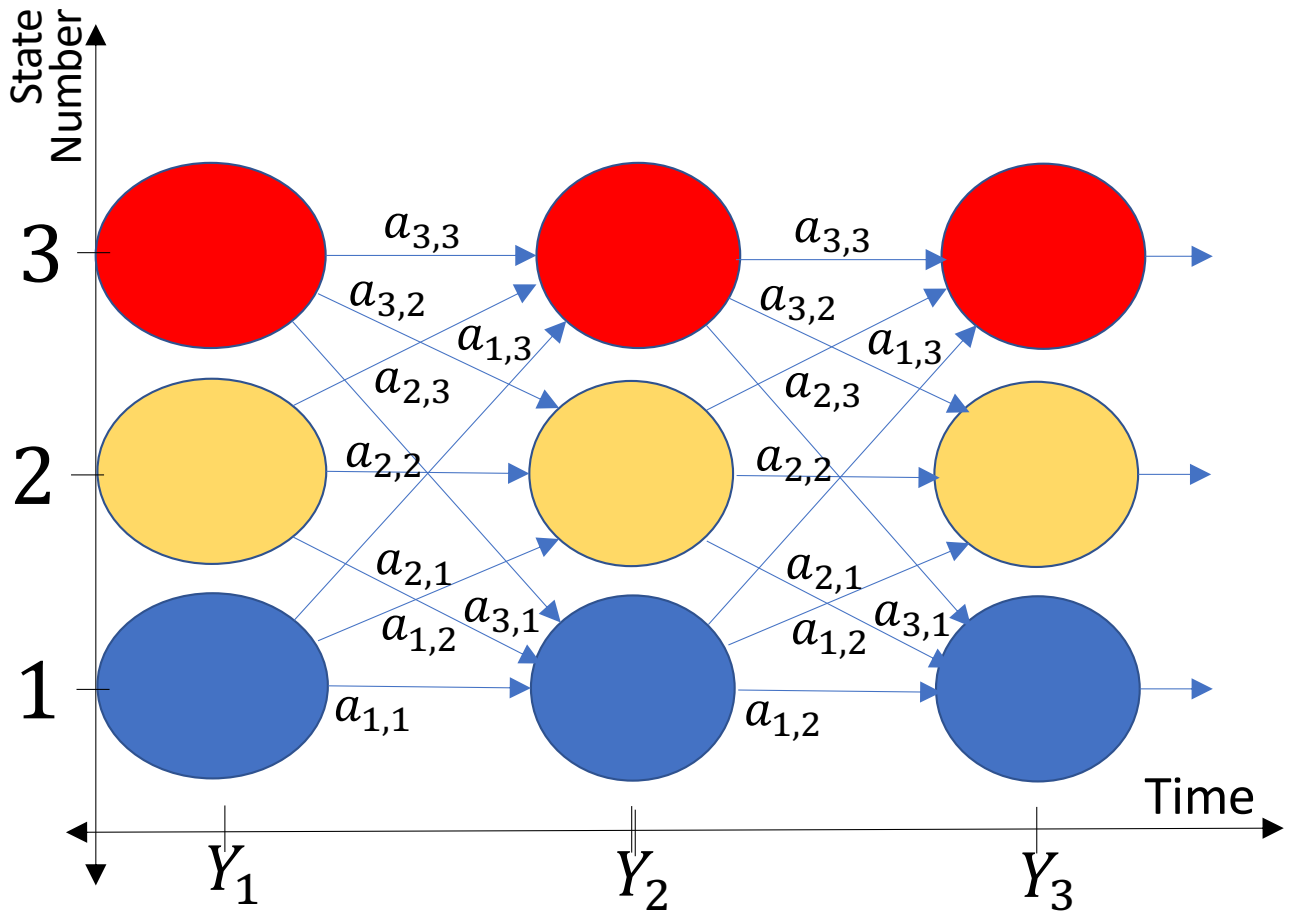
The parameters that define an HMM

- Initial State Probability:
$$\pi_i = P(Y_1 = i)$$
- Transition Probabilities:
$$a_{i,j} = P(Y_t = j | Y_{t-1} = i)$$
- Observation Probabilities:
$$b_j(x_t) = P(X_t = x_t | Y_t = j)$$



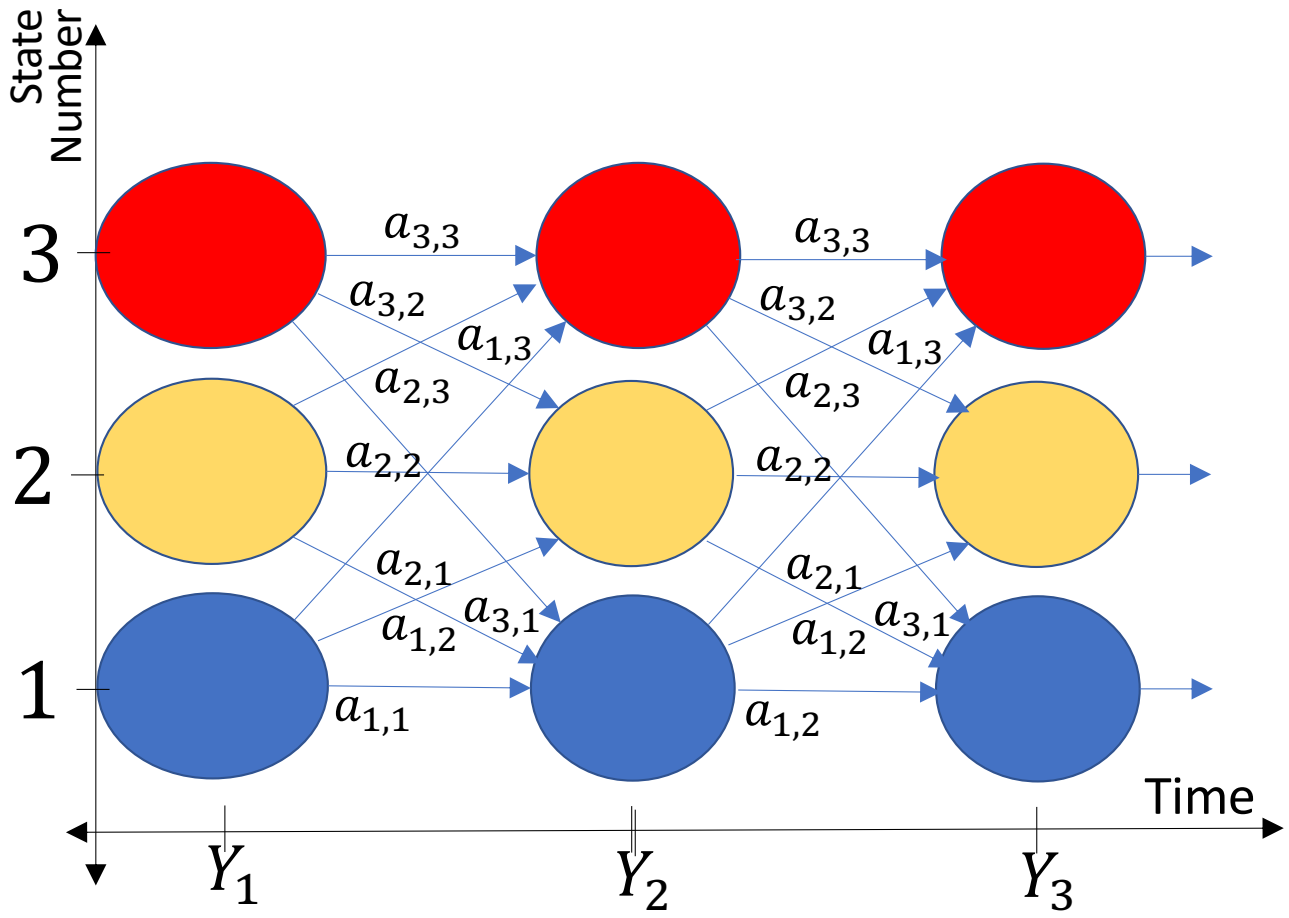
The Trellis

- Time is on the horizontal axis
- State number on the vertical axis
- Edges show state transitions: $a_{i,j}$



The Trellis

- A sequence of state variables is a path through the trellis.



For example:

$$P(Y_1 = 1, Y_2 = 3, Y_3 = 2, X_1 = \boldsymbol{x}_1, X_2 = \boldsymbol{x}_2, X_3 = \boldsymbol{x}_3) = \pi_1 b_1(\boldsymbol{x}_1) a_{1,3} b_3(\boldsymbol{x}_2) a_{3,2} b_2(\boldsymbol{x}_3)$$

Scores and backpointers

- **Score** = Probability of the best path until node j at time t

$$v_t(j) = \max_{y_1,\dots,y_{t-1}} P(Y_1 = y_1 \dots, Y_{t-1} = y_{t-1}, Y_t = j, X_0 = x_0, \dots, X_t = x_t)$$

- **Backpointer** = which node precedes node j on the best path?

$$\psi_t(j) = \operatorname*{argmax}_{y_{t-1}} \max_{y_1,\dots,y_{t-2}} P(Y_1 = y_1 \dots, Y_{t-1} = y_{t-1}, Y_t = j, X_0 = x_0, \dots, X_t = x_t)$$

Forward tracing and backtracing

- **Forward tracing** = Work from left to right through the trellis, finding the score of every node

$$v_t(j) = \max_{y_1,\dots,y_{t-1}} P(Y_1 = y_1 \dots, Y_{t-1} = y_{t-1}, Y_t = j, X_0 = x_0, \dots, X_t = x_t)$$

- **Backtracing** = Work from right to left (backward), finding the best path that ends up in a known desirable endpoint

$$\psi_t(j) = \operatorname*{argmax}_{y_{t-1}} \max_{y_1,\dots,y_{t-2}} P(Y_1 = y_1 \dots, Y_{t-1} = y_{t-1}, Y_t = j, X_0 = x_0, \dots, X_t = x_t)$$

Viterbi Algorithm

- Initialization: for all states i:

$$v_1(i) = \pi_i b_i(\boldsymbol{x}_1)$$

- Forward-Tracing:

$$v_t(j) = \max_i v_{t-1}(i)\, a_{i,j} b_j(\boldsymbol{x}_t), \qquad t = 2, \dots, d$$

$$\psi_t(j) = \operatorname{argmax}_i v_{t-1}(i)\, a_{i,j} b_j(\boldsymbol{x}_t), \qquad t = 2, \dots, d$$

- Termination:

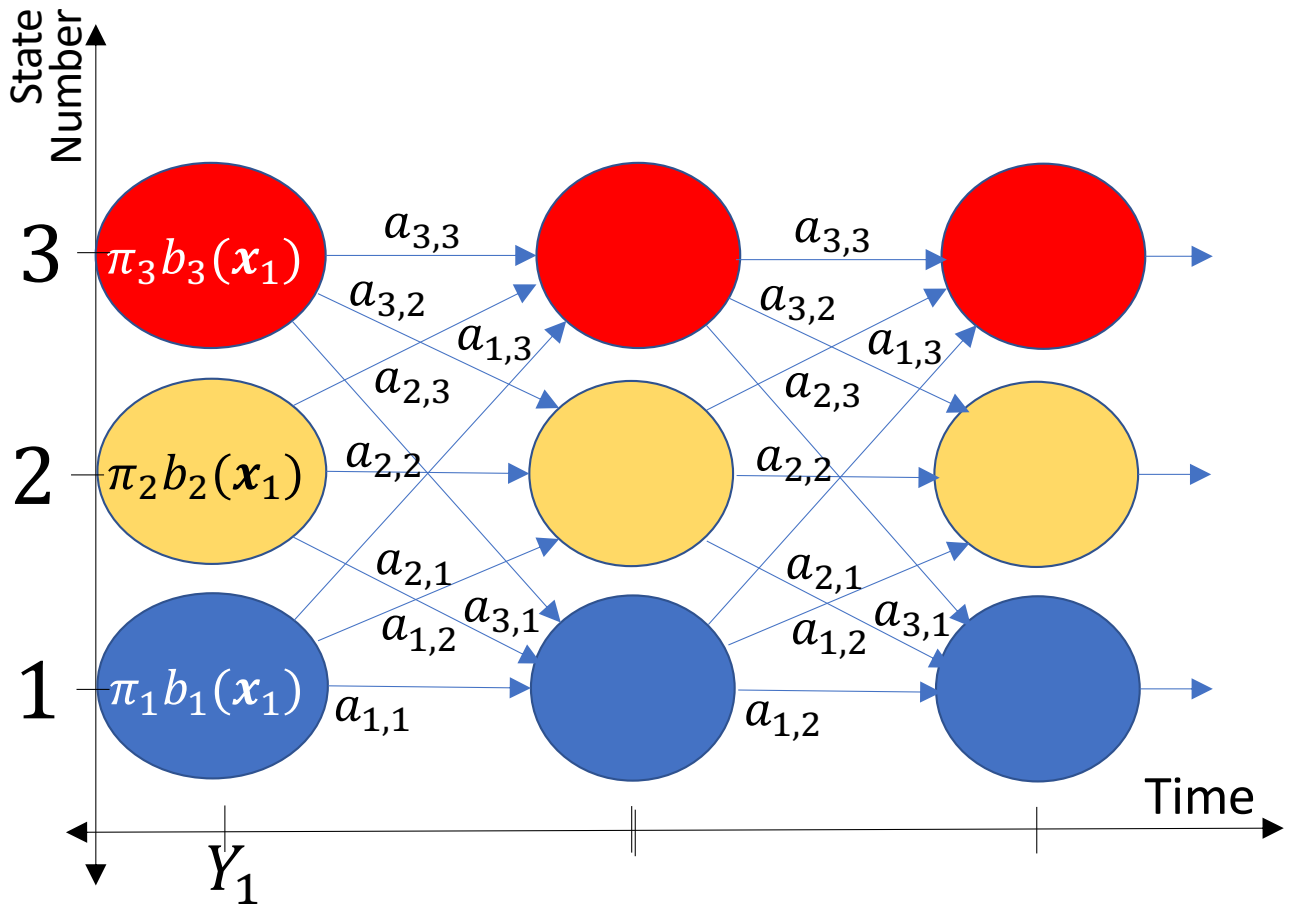
$$y_d = \operatorname{argmax}_i v_d(i)$$

- Back-Tracing:

$$y_t = \psi_{t+1}(y_{t+1}), \qquad t = d-1, \dots, 1$$

Initialization

$$v_1(i) = \pi_i b_i(\boldsymbol{x}_1)$$



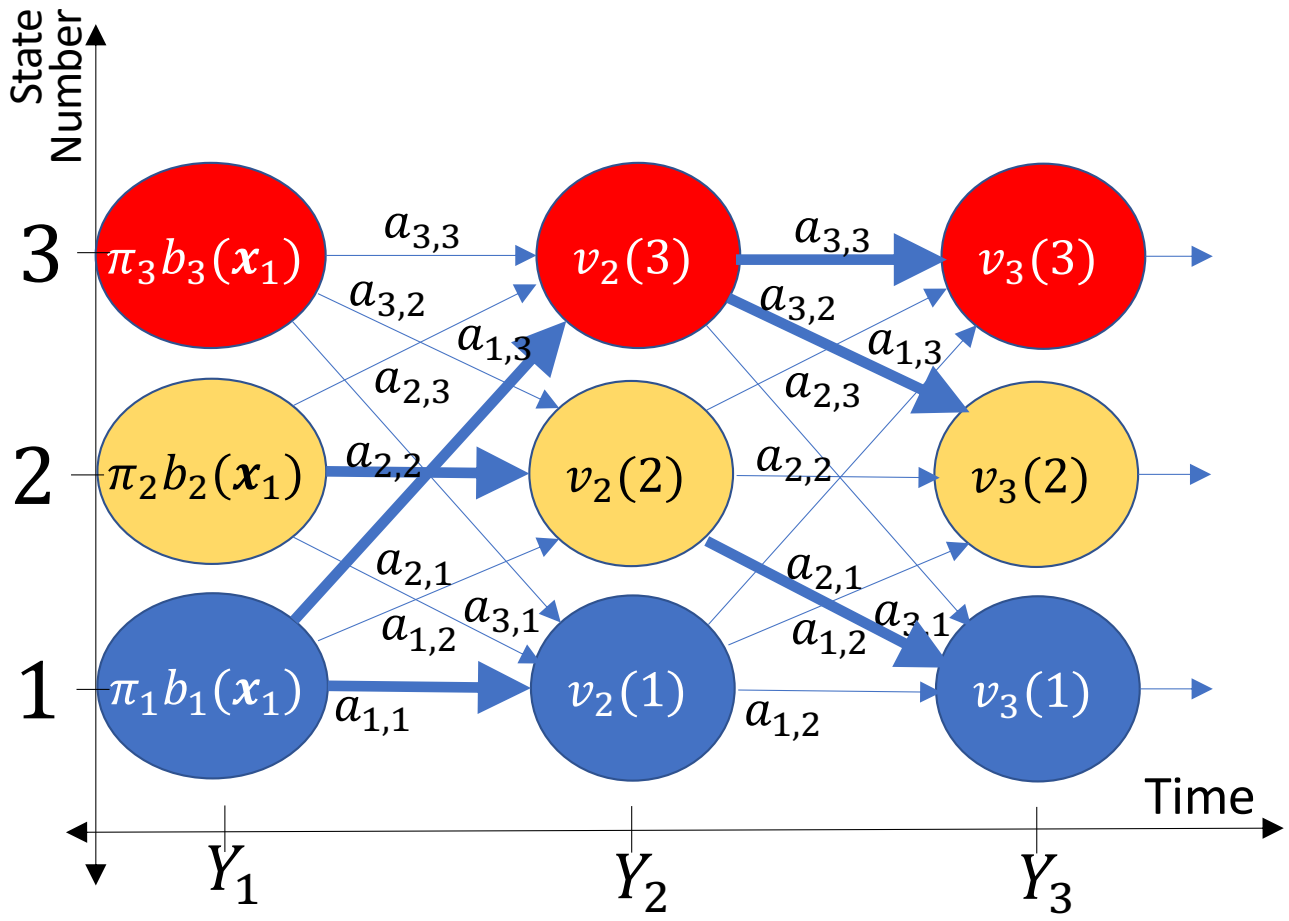
Forward-tracing

Each node now has a value:

$$v_t(j) = \max_i v_{t-1}(i)\, a_{i,j} b_j(\boldsymbol{x}_t)$$

… and there is exactly one backpointer, from every node, to exactly one node in the previous time step:

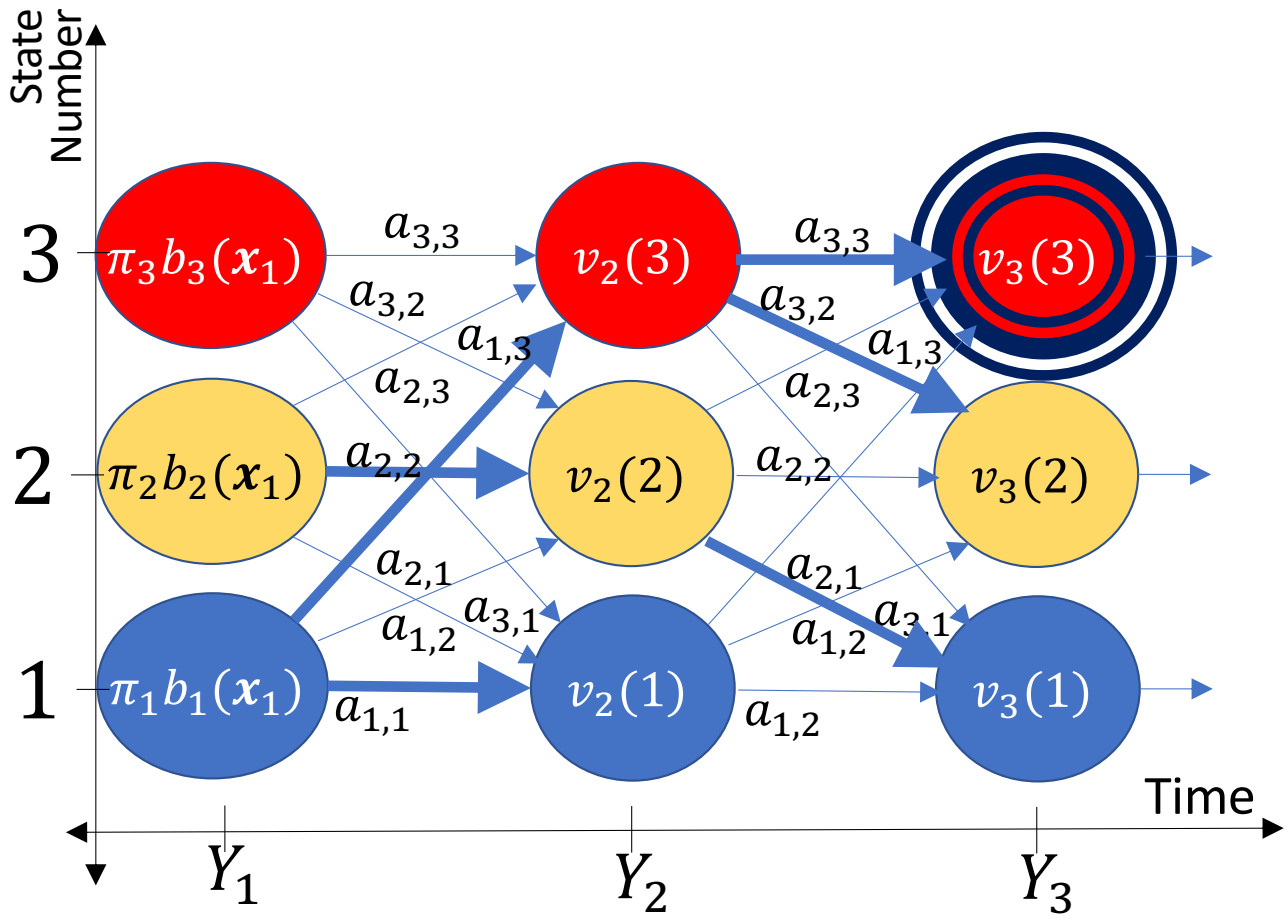
$$\psi_t(j) = \operatorname*{argmax}_i v_{t-1}(i)\, a_{i,j} b_j(\boldsymbol{x}_t)$$



Termination

The best path is the one that ends with the highest-value node:

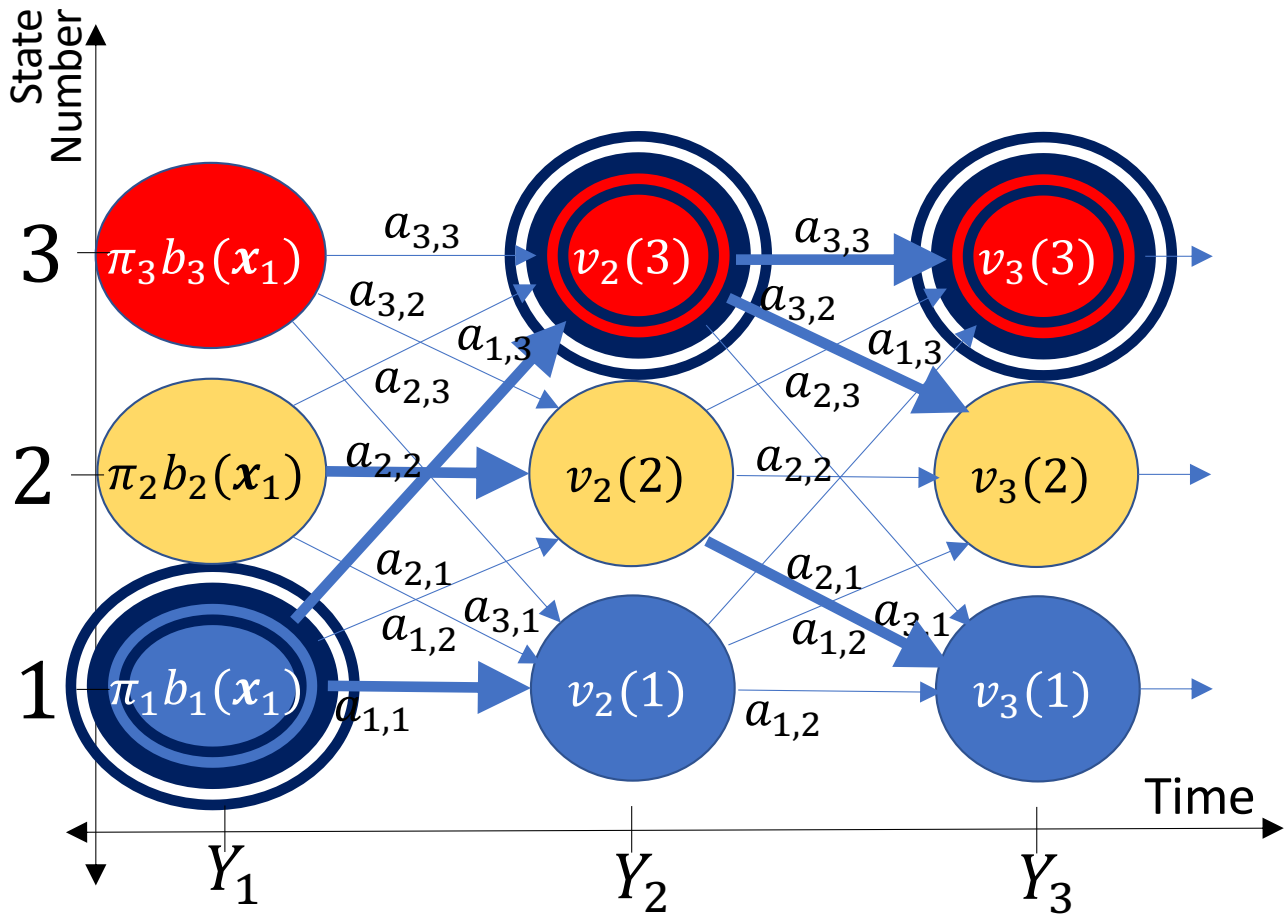
$$y_d = \underset{i}{\operatorname{argmax}}\, v_d(i)$$



Back-Tracing

The most likely state sequence is the one that ends with the highest-value node:

$$y_t = \psi_{t+1}(y_{t+1})$$



Viterbi Algorithm Computational Complexity

- Initialization: for $i \in \mathcal{Y}$:

$$v_1(i) = \pi_i b_i(\boldsymbol{x}_1)$$

$\mathcal{O}\{|\mathcal{Y}|\}$

- Iteration: for $2 \leq t \leq d$, for $j \in \mathcal{Y}$:

$$v_t(j) = \max_{i \in \mathcal{Y}} v_{t-1}(i)\, a_{i,j} b_j(x_t)$$

$$\psi_t(j) = \operatorname*{argmax}_{i \in \mathcal{Y}} v_{t-1}(i)\, a_{i,j} b_j(x_t)$$

$\mathcal{O}\{d|\mathcal{Y}|^2\}$

- Termination:

$$y_d = \operatorname*{argmax}_{i \in \mathcal{Y}} v_d(i)$$

$\mathcal{O}\{|\mathcal{Y}|\}$

- Back-Trace:

$$y_t = \psi_{t+1}(y_{t+1})$$

$\mathcal{O}\{d\}$

Total: $\mathcal{O}\{d|\mathcal{Y}|^2\}$

Try the quiz!

- Go to prairielearn, try the quiz!

Outline

- Review: Bayesian classifer, Bayesian networks

$$f(x) = \underset{y}{\operatorname{argmax}}\, P(Y = y | X = x)$$

- HMM: Probabilistic reasoning over time

$$\pi_i = P(Y_1 = i)$$
$$a_{i,j} = P(Y_t = j | Y_{t-1} = i)$$
$$b_j(\boldsymbol{x}_t) = P(X_t = \boldsymbol{x}_t | Y_t = j)$$

- Viterbi algorithm

$$v_t(j) = \max_{i \in \mathcal{Y}} v_{t-1}(i)\, a_{i,j} b_j(\boldsymbol{x}_t)$$
$$\psi_t(j) = \underset{i \in \mathcal{Y}}{\operatorname{argmax}}\, v_{t-1}(i)\, a_{i,j} b_j(\boldsymbol{x}_t)$$