Plan of the Lecture

- Review: Proportional-Integral-Derivative (PID) control
- Today's topic: introduction to Root Locus design method

Goal: introduce the Root Locus method as a way of visualizing the locations of closed-loop poles of a given system as some parameter is varied.

Reading: FPE, Chapter 5

Note!! The way I teach the Root Locus differs a bit from what the textbook does (good news: it is simpler). Still, pay attention in class!!

Course structure so far:

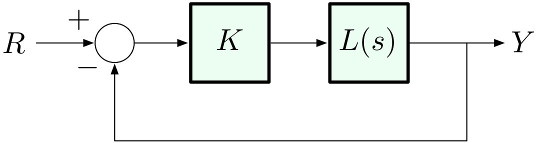
modeling	—	examples
$\downarrow$		
analysis	—	transfer function, response, stability
$\downarrow$		
design	—	some simple examples given

We will focus on design from now on.

The Root Locus Design Method

(invented by Walter R. Evans in 1948)

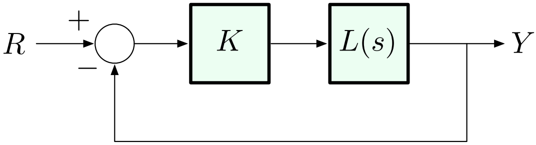
Consider this unity feedback configuration:



where

- K is a constant gain
- $L(s) = \dfrac{b(s)}{a(s)}$, where $a(s)$ and $b(s)$ are some polynomials

The Root Locus Design Method



Closed-loop transfer function: $\dfrac{Y}{R} = \dfrac{KL(s)}{1+KL(s)}, \; L(s) = \dfrac{b(s)}{a(s)}$

Closed loop poles are solutions of:

$$1 + KL(s) = 0 \qquad \Leftrightarrow \qquad L(s) = -\frac{1}{K}$$

$$\Updownarrow$$

$$1 + \frac{Kb(s)}{a(s)} = 0$$

$$\Updownarrow$$

$$\underbrace{a(s) + Kb(s)}_{\substack{\text{characteristic} \\ \text{polynomial}}} = 0 \qquad \text{characteristic equation}$$

A Comment on Change of Notation

Note the change of notation:

$$\text{from } H(s) \text{ or } G(s) = \frac{q(s)}{p(s)} \quad \text{to } L(s) = \frac{b(s)}{a(s)}$$

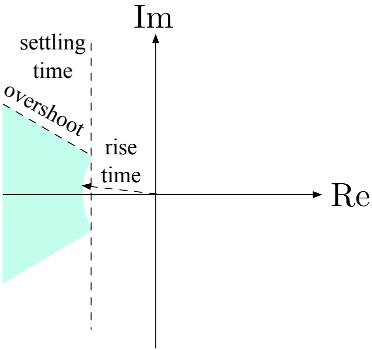
— the RL method is quite general, so $L(s)$ is not necessarily the *plant* transfer function, and K is not necessary *feedback gain* (could be *any parameter*).

E.g., $L(s)$ and K may be related to plant transfer function and feedback gain through some transformation.

As long as we can represent the poles of the closed-loop transfer function as roots of the equation $1 + KL(s) = 0$ for *some choice* of K and $L(s)$, we can apply the RL method.

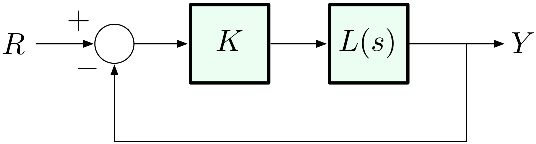
Towards Quantitative Characterization of Stability

Qualitative description of stability: Routh test gives us a range of K to guarantee stability.



For what values of K do we best satisfy given design specs?

Root Locus and Quantitative Stability



Closed-loop transfer function: $\frac{Y}{R} = \frac{KL(s)}{1+KL(s)}, \; L(s) = \frac{b(s)}{a(s)}$

For what values of K do we best satisfy given design specs?

Specs are encoded in pole locations, so:

The *root locus* for $1 + KL(s)$ is the set of all closed-loop poles, i.e., the roots of

$$1 + KL(s) = 0,$$

as K varies from 0 to ∞.

A Simple Example

$$L(s) = \frac{1}{s^2 + s} \qquad b(s) = 1, \; a(s) = s^2 + s$$

Characteristic equation: $\qquad a(s) + Kb(s) = 0$

$$s^2 + s + K = 0$$

Here, we can just use the quadratic formula:

$$s = -\frac{1 \pm \sqrt{1 - 4K}}{2} = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2}$$

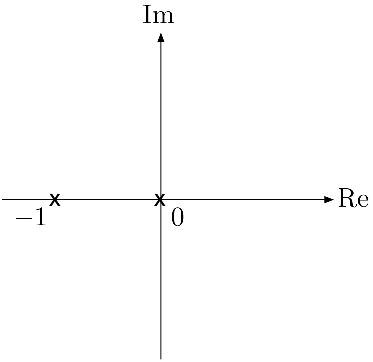
$$\text{Root locus} = \left\{ -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} \; : 0 \leq K < \infty \right\} \subset \mathbb{C}$$

Example, continued

$$\text{Root locus} = \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \;:\; 0 \le K < \infty \right\} \subset \mathbb{C}$$

Let's plot it in the s-plane:

- start at $K = 0$ the roots are $-\frac{1}{2} \pm \frac{1}{2} \equiv -1, 0$
  note: these are poles of L (open-loop poles)

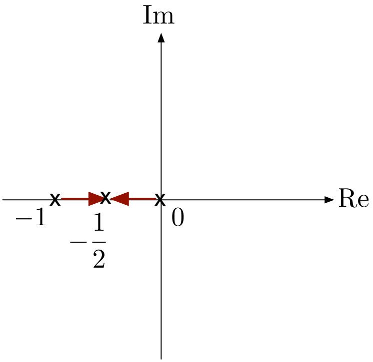


Example, continued

$$\text{Root locus: } \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \ : 0 \le K < \infty \right\} \subset \mathbb{C}$$

- as K increases from 0, the poles start to move

$$\begin{aligned} 1 - 4K > 0 \qquad &\Longrightarrow \ 2 \text{ real roots} \\ K = 1/4 \qquad &\Longrightarrow 1 \text{ real root } s = -1/2 \end{aligned}$$

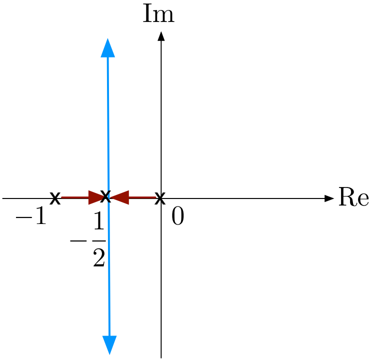


Example, continued

$$\text{Root locus: } \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \; : 0 \le K < \infty \right\} \subset \mathbb{C}$$

- as K increases from 0, the poles start to move

$$K > 1/4 \qquad \Longrightarrow \; 2 \text{ complex roots with } \mathrm{Re}(s) = -1/2$$

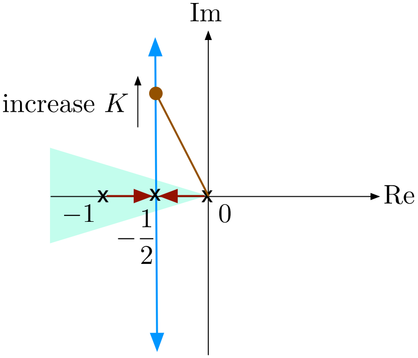


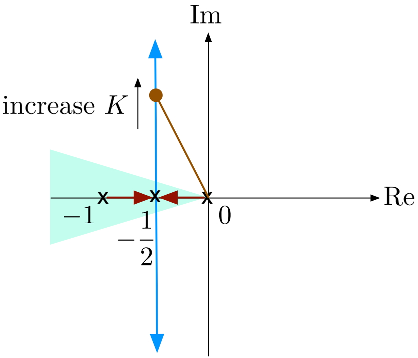
($s = -1/2$ is the *point of breakaway* from the real axis)

Example, continued

Compare this to admissible regions for given specs:

$t_s \approx \dfrac{3}{\sigma}$	want σ large, can only have $\sigma = \dfrac{1}{2}$ $(t_s = 6)$
$t_r \approx \dfrac{1.8}{\omega_n}$	want ω_n large $\Longrightarrow$ want K large
M_p	want to be inside the shaded region $\Longrightarrow$ want K small





Thus, the root locus helps us *visualize the trade-off* between all the specs in terms of K.

However, for order > 2, there will generally be no direct formula for the closed-loop poles as a function of K.

Our goal: develop simple rules for (approximately) sketching the root locus in the general case.

Equivalent Characterization of RL: Phase Condition

Recall our original definition: The *root locus* for $1 + KL(s)$ is the set of all closed-loop poles, i.e., the roots of

$$1 + KL(s) = 0,$$

as K varies from 0 to ∞.

A point $s \in \mathbb{C}$ is on the RL if and only if

$$L(s) = \underbrace{-\frac{1}{K}}_{\text{negative and real}} \qquad \text{for some } K > 0$$

This gives us an equivalent characterization:

The phase condition: The root locus of $1 + KL(s)$ is the set of all $s \in \mathbb{C}$, such that $\angle L(s) = 180^\circ$, i.e., $L(s)$ is real and negative.

Six Rules for Sketching Root Loci

There are *six rules* for sketching root loci. These rules are mainly qualitative, and their purpose is to give intuition about impact of poles and zeros on performance.

These rules are:

- Rule A — number of branches
- Rule B — start points
- Rule C — end points
- Rule D — real locus
- Rule E — asymptotes
- Rule F — $j\omega$-crossings

Today, we will cover mostly Rules A–C (and a bit of D).

Rule A: Number of Branches

$$1 + K\frac{b(s)}{a(s)} = 1 + K\frac{s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} = 0$$

$$\Longrightarrow (s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n) + K(s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m) = 0$$

Since $\deg(a) = n \geq m = \deg(b)$, the characteristic polynomial $a(s) + Kb(s) = 0$ has degree n.

The characteristic polynomial has n solutions (roots), some of which may be repeated. As we vary K, these n solutions also vary to form n branches.

Rule A:

$$\#(\text{branches}) = \deg(a)$$

Rule B: Start Points

The locus starts from $K = 0$. What happens near $K = 0$?

If $a(s) + Kb(s) = 0$ and $K \sim 0$, then $a(s) \approx 0$.

Therefore:

- s is close to a root of $a(s) = 0$, or
- s is close to a pole of $L(s)$

Rule B: branches start at open-loop poles.

Rule C: End Points

What happens to the locus as $K \to \infty$?

$$a(s) + Kb(s) = 0$$
$$b(s) = -\frac{1}{K}a(s)$$

— as $K \to \infty$,

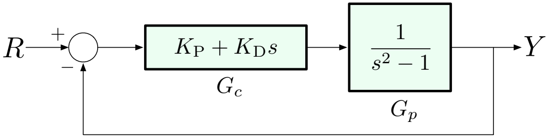
- branches end at the roots of $b(s) = 0$, or
- branches end at zeros of $L(s)$

Rule C: branches end at open-loop zeros.

Note: if $n > m$, we have n branches, but only m zeros. The remaining $n - m$ branches go off to infinity (end at "zeros at infinity").

Example

PD control of an unstable 2nd-order plant



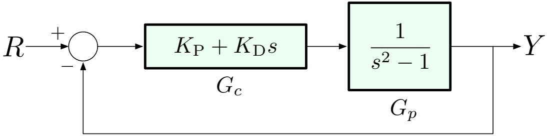
$$\frac{Y}{R} = \frac{G_c G_p}{1 + G_c G_p} \qquad \text{poles: } 1 + G_c(s) G_p(s) = 0$$

$$1 + (K_\mathrm{P} + K_\mathrm{D} s)\left(\frac{1}{s^2 - 1}\right) = 0$$

We will examine the impact of varying $K = K_\mathrm{D}$, assuming the ratio $K_\mathrm{P}/K_\mathrm{D}$ *fixed*.

Example

PD control of an unstable 2nd-order plant



We will examine the impact of varying $K = K_D$, assuming the ratio K_P/K_D *fixed*.

Let us write the characteristic equation in *Evans form*:

$$1 + \underbrace{K_D}_{K}\left(s + \frac{K_P}{K_D}\right)\left(\frac{1}{s^2-1}\right) = 1 + K\underbrace{\frac{s + K_P/K_D}{s^2-1}}_{L(s)} = 0$$

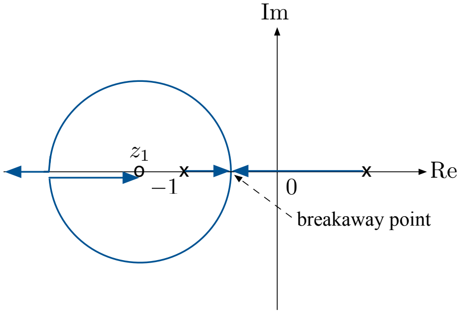
$$L(s) = \frac{s - z_1}{s^2 - 1} \qquad \text{zero at } s = z_1 = -K_P/K_D < 0$$

Example

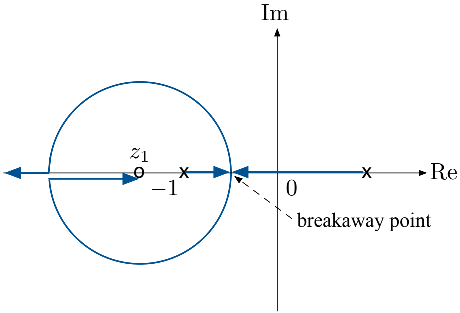
$$L(s) = \frac{s - z_1}{s^2 - 1}$$

- Rule A: $\begin{cases} m = 1 \\ n = 2 \end{cases} \implies 2$ branches
- Rule B: branches start at open-loop poles $\quad s = \pm 1$
- Rule C: branches end at open-loop zeros $\quad s = z_1, -\infty$
  (we will see why $-\infty$ later)

So the root locus will look something like this:



$$L(s) = \frac{s - z_1}{s^2 - 1}$$



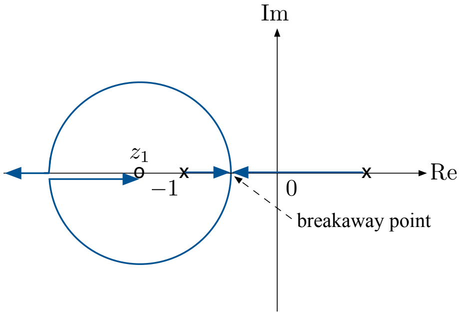
Why does one of the branches go off to $-\infty$?

$s^2 - 1 + K(s - z_1) = 0$

$s^2 + Ks - (Kz_1 + 1) = 0$

$s = -\frac{K}{2} \pm \sqrt{\frac{K^2}{4} + Kz_1 + 1}, \; z_1 < 0$ as $K \to \infty$, s will be < 0

$$L(s) = \frac{s - z_1}{s^2 - 1}$$



Is the point $s = 0$ on the root locus?

Let's see if there is any value $K > 0$, for which this is possible:

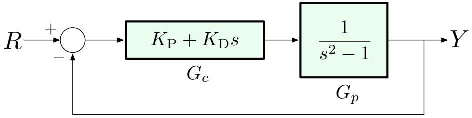
$$1 + KL(0) = 0$$

$$1 + Kz_1 = 0 \qquad K = -\frac{1}{z_1} > 0 \text{ does the job}$$

From Root Locus to Time Response Specs

For concreteness, let's see what happens when

$$K_{\mathrm{P}}/K_{\mathrm{D}} = -z_1 = 2 \qquad \text{and} \qquad K = K_{\mathrm{D}} = 5 \Longrightarrow K_{\mathrm{P}} = 10$$



$$G_c(s) = 10 + 5s$$

$$u = 10e + 5\dot{e}, \qquad e = r - y$$

$$\text{Characteristic equation:} \quad 1 + 5\left(\frac{s+2}{s^2-1}\right) = 0$$

$$s^2 + 5s + 9 = 0$$

Relate to 2nd-order response: $\omega_n^2 = 9,\ 2\zeta\omega_n = 5 \Longrightarrow \zeta = 5/6$

Main Points

- When zeros are in LHP, *high gain* can be used to stabilize the system (although one must worry about zeros at infinity).
- If there are zeros in RHP, high gain is always disastrous.
- PD control is effective for stabilization because it introduces a zero in LHP.

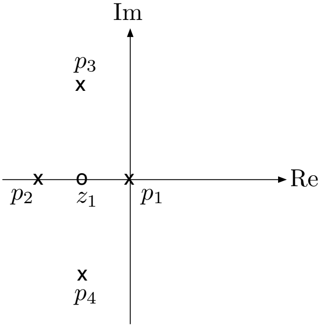
But: Rules A–C cannot tell the whole story. How do we know which way the branches go, and which pole corresponds to which zero?

Rules D–F!!

Example

Let's consider $\quad L(s) = \dfrac{s+1}{s(s+2)\big((s+1)^2+1\big)}$

- Rule A: $\begin{cases} m = 1 \\ n = 4 \end{cases} \quad \Longrightarrow$ 4 branches
- Rule B: branches start at open-loop poles
  $s = 0,\ s = -2,\ s = -1 \pm j$
- Rule C: branches end at open-loop zeros $\qquad s = -1, \pm\infty$



Example, continued

Three more rules:

- Rule D: real locus
- Rule E: asymptotes
- Rule F: $j\omega$-crossings

Rules D and E are both based on the fact that

$$1 + KL(s) = 0 \text{ for some } K > 0 \quad \Longleftrightarrow \quad L(s) < 0$$

Rule D: Real Locus

The branches of the RL start at the open-loop poles. Which way do they go, left or right?

Recall the phase condition:

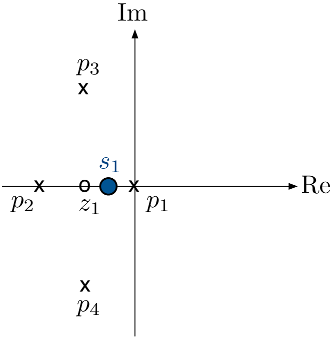
$$1 + KL(s) = 0 \qquad \Longleftrightarrow \qquad \angle L(s) = 180^\circ$$

$$\begin{aligned}
\angle L(s) &= \angle \frac{b(s)}{a(s)} \\
&= \angle \frac{(s - z_1)(s - z_2)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_n)} \\
&= \sum_{i=1}^{m} \angle(s - z_i) - \sum_{j=1}^{n} \angle(s - p_j)
\end{aligned}$$

— this sum must be $\pm 180^\circ$ for *any* s that lies on the RL.

Rule D: Real Locus

So, we try test points:



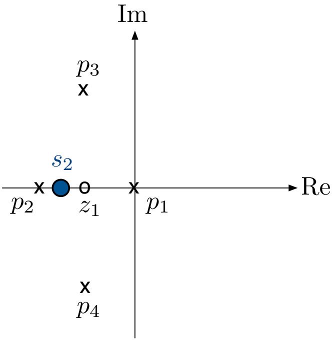
$$\angle(s_1 - z_1) = 0^\circ \quad (s_1 > z_1)$$
$$\angle(s_1 - p_1) = 180^\circ \quad (s_1 < p_1)$$
$$\angle(s_1 - p_2) = 0^\circ \quad (s_1 > p_2)$$
$$\angle(s_1 - p_3) = -\angle(s_1 - p_4)$$

(conjugate poles cancel)

$$\angle(s_1 - z_1) - [\angle(s_1 - p_1) + \angle(s_1 - p_2) + \angle(s_1 - p_3) + \angle(s_1 - p_4)]$$
$$= 0^\circ - [180^\circ + 0^\circ + 0^\circ] = -180^\circ \qquad \checkmark s_1 \text{ is on RL}$$

Rule D: Real Locus

Try more test points:



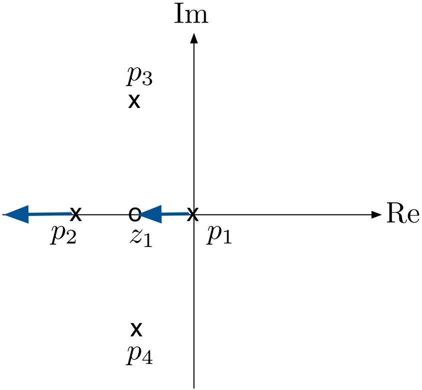
$$\angle(s_2 - z_1) = 180^\circ \quad (s_2 < z_2)$$
$$\angle(s_2 - p_1) = 180^\circ \quad (s_2 < p_1)$$
$$\angle(s_2 - p_2) = 0^\circ \quad (s_2 > p_2)$$
$$\angle(s_2 - p_3) = -\angle(s_1 - p_4)$$

(conjugate poles cancel)

$$\angle(s_2 - z_1) - [\angle(s_2 - p_1) + \angle(s_2 - p_2) + \angle(s_2 - p_3) + \angle(s_2 - p_4)]$$
$$= 180^\circ - [180^\circ + 0^\circ + 0^\circ] = 0^\circ \qquad \times \; s_1 \text{ is not on RL}$$

Rule D: Real Locus

Rule D: If s is *real*, then it is on the RL of $1 + KL$ if and only if there are an odd number of *real open-loop poles and zeros* to the right of s.



We will cover Rules E and F, and complete the RL for this example, in the next lecture.