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Reading: FPE, Sections 3.5–3.6

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$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s=-a} H_2(s) = H_1(s) + \frac{1}{a} \cdot sH_1(s)$$

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$$y_2(t) = \mathscr{L}^{-1}\{Y_2(s)\} = \mathscr{L}^{-1}\left\{Y_1(s) + \frac{1}{a} \cdot sY_1(s)\right\} = y_1(t) + \frac{1}{a}\dot{y}_1(t)$$

(assuming zero initial conditions)

Effect of a LHP Zero

Step response (zero at $s = -a$)

$$y_2(t) = y_1(t) + \frac{1}{a}\dot{y}_1(t) \qquad \text{where } y_1(t) = \text{original step response}$$

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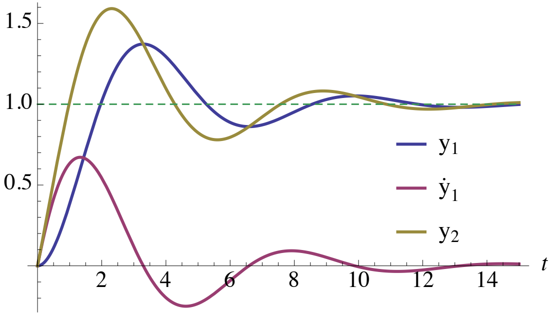

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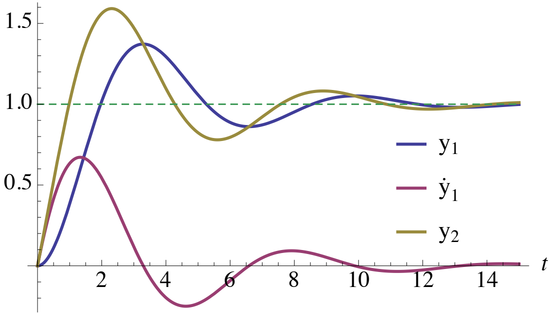

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- increased overshoot (major effect)
- little influence on settling time
- what happens as $a \to \infty$?

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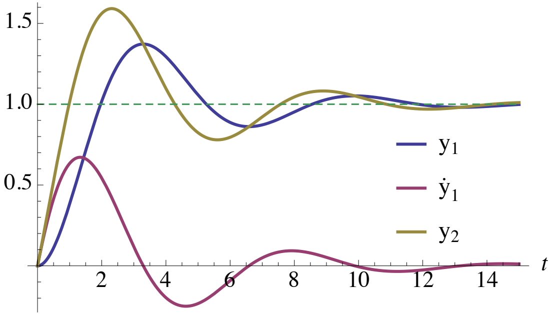

Effects of a LHP zero:

- increased overshoot (major effect)
- little influence on settling time
- what happens as $a \to \infty$? — effects become less significant

What About a RHP Zero?

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s = a} H_2(s) = H_1(s) - \frac{1}{a} \cdot s H_1(s)$$

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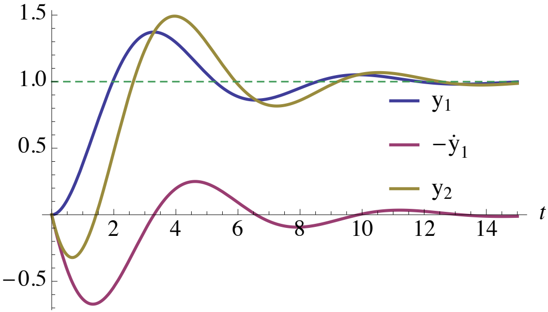

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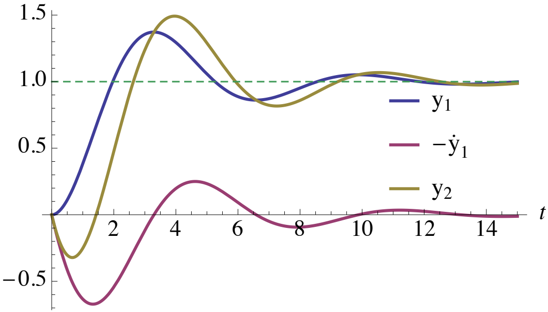
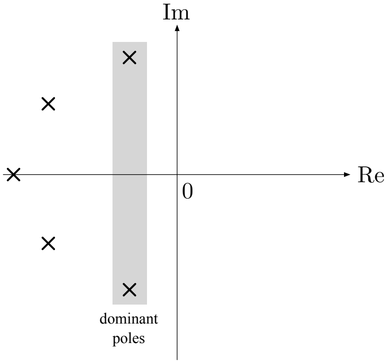


Effects of a RHP zero:

- slows down (delays) the response
- creates *undershoot* (at least, when a is small enough)

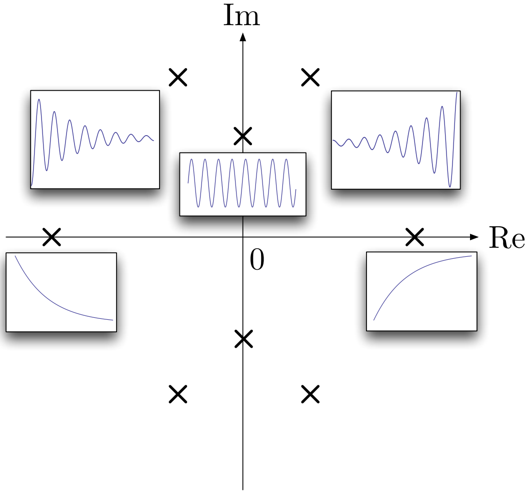
Effect of Extra Poles

A general nth-order system has n poles



- extra LHP poles are not significant if their real parts are at least $5\times$ the real parts of dominant LHP poles
- e.g., if dominant poles have $\mathrm{Re}(s) = -2$ and we have extra poles with $\mathrm{Re}(s) = -10$, their time-domain contributions will be e^{-2t} and $e^{-10t} \ll e^{-2t}$
- $5\times$ is just a convention, but we can really see the effect of extra poles that are closer (cf. Lab 2)

Effect of Pole Locations



- poles in open LHP ($\mathrm{Re}(s) < 0$) — stable response
- poles in open RHP ($\mathrm{Re}(s) > 0$) — unstable response
- poles on the imaginary axis ($\mathrm{Re}(s) = 0$) — tricky case

Marginal Case: Poles on the Imaginary Axis

Let's consider the case of a pole at the origin: $H(s) = \dfrac{1}{s}$

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- impulse response: $Y(s) = \dfrac{1}{s} \Longrightarrow y(t) = 1(t)$ (OK)
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- impulse response: $Y(s) = \dfrac{\omega^2}{s^2 + \omega^2} \Longrightarrow y(t) = \omega \sin(\omega t)$
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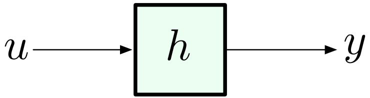
Systems with poles on the imaginary axis are *not stable*.

What Is Stability?



One reasonable definition is as follows:

A linear time-invariant system is *Bounded-Input, Bounded-Output* (BIBO) *stable* provided either one of the following three equivalent conditions is satisfied:

1. If every bounded input $u(t)$ results in a bounded output $y(t)$, regardless of initial conditions.
2. If the impulse response $h(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)| \, \mathrm{d}t < \infty.$$

3. If all poles of the transfer function $H(s)$ are *strictly stable* (lie in open LHP).

Checking for Stability?

Consider a general transfer function:

$$H(s) = \frac{q(s)}{p(s)}$$

where q and p are polynomials, and $\deg(q) \leq \deg(p)$.

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Now, this is hard to do for high-degree polynomials — it's computationally intensive, especially symbolically.

But: often we *don't need to know* precise pole locations, just need to know that they are strictly stable.

Checking for Stability

Problem: given an nth-degree polynomial

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

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Terminology: we often say that the polynomial p is (strictly) stable if all of its roots are.

A Necessary Condition for Stability

Terminology: we say that A is a necessary condition for B if

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Proof: suppose that p has roots at $r_1, r_2, \dots, r_n$ with $\mathrm{Re}(r_i) < 0$ for all i. Then

$$p(s) = (s - r_1)(s - r_2) \dots (s - r_n)$$

— multiply this out and check that all coefficients are positive.

Routh–Hurwitz Criterion

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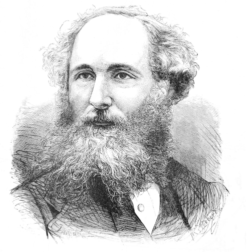
We will now introduce a necessary and sufficient condition for stability: the *Routh–Hurwitz Criterion*.

Routh–Hurwitz Criterion: A Bit of History

J.C. Maxwell, "On governors," Proc. Royal Society, no. 100, 1868

... [Stability of the governor] is mathematically equivalent to the condition that all the possible roots, and all the possible parts of the impossible roots, of a certain equation shall be negative. ...

I have not been able completely to determine these conditions for equations of a higher degree than the third; but I hope that the subject will obtain the attention of mathematicians.



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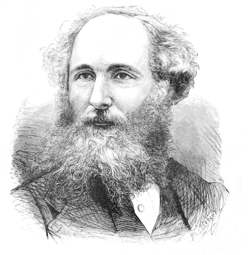

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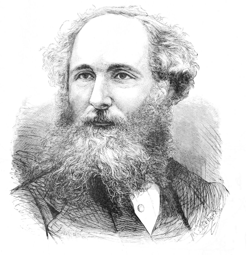

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In 1893, Adolf Hurwitz solved the same problem, using a different method, independently of Routh.

Edward John Routh, 1831–1907

Adolf Hurwitz, 1859–1919

Routh's Test

Problem: check whether the polynomial

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

is strictly stable.

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We begin by forming the Routh array using the coefficients of p:

$$\begin{array}{rccccc} s^n: & 1 & a_2 & a_4 & a_6 & \ldots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \ldots \end{array}$$

(if necessary, add zeros in the second row to match lengths)

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Note that the very first entry is always 1, and also note the order in which the coefficients are filled in.

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Next, we form the third row marked by s^{n-2}:

$$s^{n-2}: \quad b_1 \quad b_2 \quad b_3 \quad \dots$$

where

Routh's Test

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Next, we form the third row marked by s^{n-2}:

$$s^{n-2}: \quad b_1 \quad b_2 \quad b_3 \quad \dots$$

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Note: the new row is 1 element shorter than the one above it

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Eventually, we complete the array like this:

$$\begin{array}{rllllll} s^n: & 1 & a_2 & a_4 & a_6 & \dots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \dots \\ s^{n-2}: & b_1 & b_2 & b_3 & \dots \\ s^{n-3}: & c_1 & c_2 & \dots \\ \vdots \\ s^1: & * & * \\ s^0: & * \end{array}$$

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s^1: & * & * & & & \\
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\end{array}$$

(as long as we don't get stuck with division by zero: more on this later)

After the process terminates, we will have $n+1$ entries in the first column.

The Routh–Hurwitz Criterion

Consider degree-n polynomial

$$p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

and form the Routh array:

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The Routh–Hurwitz criterion: Assume that the necessary condition for stability holds, i.e., $a_1, \ldots, a_n > 0$. Then:

- p is stable if and only if all entries in the first column are positive;
- otherwise, #(RHP poles) = #(sign changes in 1st column)

Example

Check stability of

$$p(s) = s^4 + 4s^3 + s^2 + 2s + 3$$

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Answer: p is unstable — it has 2 RHP poles (2 sign changes in 1st column)

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— p is stable iff $a_1, a_2, a_3 > 0$ (necc. cond.) and $a_1 a_2 > a_3$

Stability Conditions for Low-Order Polynomials

The upshot:

- A 2nd-degree polynomial $p(s) = s^2 + a_1 s + a_2$ is stable if and only if $a_1 > 0$ and $a_2 > 0$
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- These conditions were already obtained by Maxwell in 1868.
- In both cases, the computations were *purely symbolic*: this can make a lot of difference in *design*, as opposed to *analysis*.

Routh–Hurwitz as a Design Tool

Parametric stability range

We can use the Routh test to determine *parameter ranges* for stability.

Routh–Hurwitz as a Design Tool

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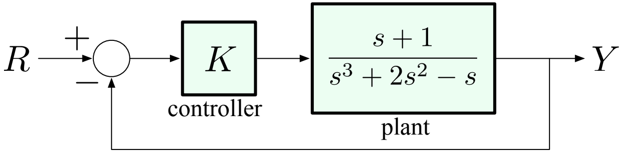
Example: consider the unity feedback configuration



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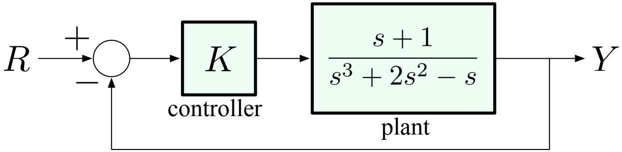

Note that the plant is *unstable* (the denominator has a negative coefficient and a zero coefficient).

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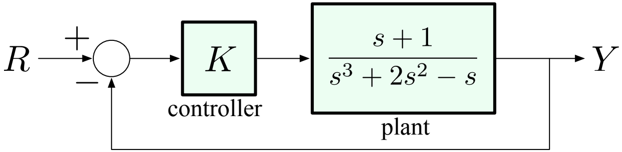

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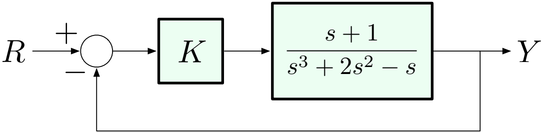
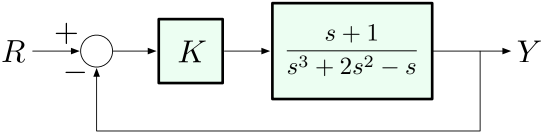
Problem: determine the range of values the scalar gain K can take, for which the closed-loop system is stable.

Example, continued



Problem: determine the range of values the scalar gain K can take, for which the closed-loop system is stable.

Example, continued



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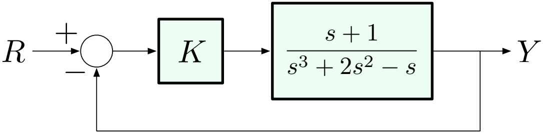

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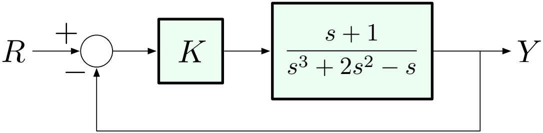

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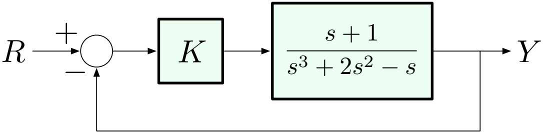

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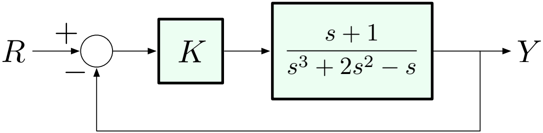

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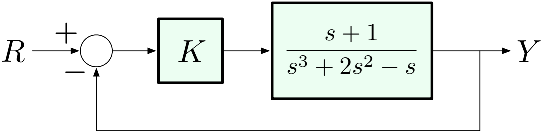

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Now we need to test stability of $p(s) = s^3 + 2s^2 + (K-1)s + K$.

Example, continued

Test stability of

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using the Routh test.

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Note: The necessary condition requires $K > 1$, but now we actually know that we must have $K > 2$ for stability.

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- For an *entire row of zeros*, the procedure is a more complicated (see Example 3.34 in FPE) – we will not worry about this too much.