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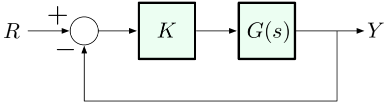
Reading: FPE, Section 6.1

Stability from Frequency Response

Consider this unity feedback configuration:



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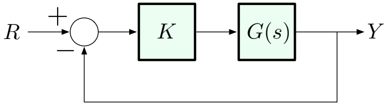
Stability from Frequency Response



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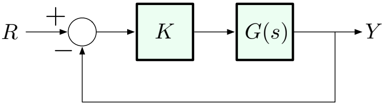

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$$1 + KG(s) = 0 \qquad \Longleftrightarrow \qquad KG(s) = -1 \qquad \left(\Longleftrightarrow G(s) = -\frac{1}{K} \right)$$

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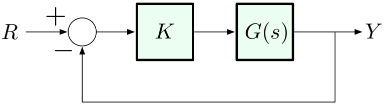

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If $s \in \mathbb{C}$ is on the RL, then

$$|KG(s)| = 1 \qquad \text{and} \qquad \angle KG(s) = \angle G(s) = 180^\circ \mod 360^\circ$$

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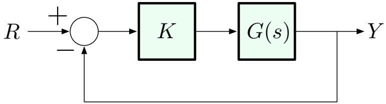

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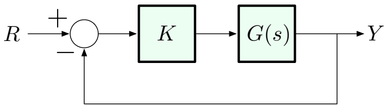

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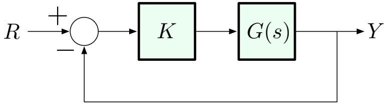

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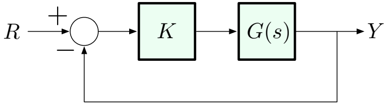

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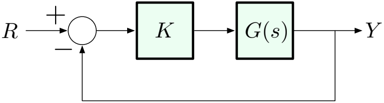
How does this relate to the root locus? $j\omega$-crossings!!

Stability from Frequency Response



Stability from frequency response. If $s = j\omega$ is on the root locus (for some value of K), then

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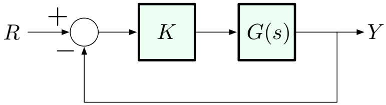

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Therefore, the transition from stability to instability can be detected in two different ways:

- from root locus — as $j\omega$-crossings
- from Bode plots — as $M = 1$ and $\phi = 180^\circ$ at some frequency ω (for a given value of K)

Example

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Recall the necessary & sufficient condition for stability for a 3rd-degree polynomial $s^3 + a_1 s^2 + a_2 s + a_3$:

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Let's see what we can read off from the Bode plots.

Example, continued

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Bode form: $$KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

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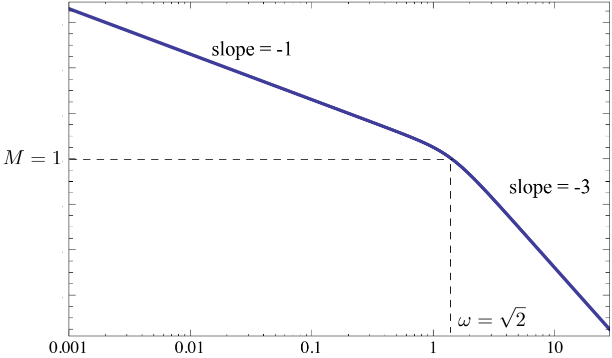
- $\zeta = \dfrac{1}{\sqrt{2}} \implies$ no reasonant peak

Example, Magnitude Plot

$$KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Magnitude plot for $K = 4$ (the critical value):



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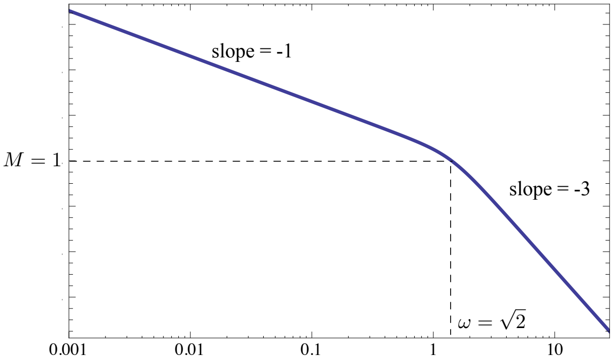

When $\omega = \sqrt{2}$, $M = |4G(j\omega)| = \left|\frac{2}{j\sqrt{2}\left(j^2 + j\sqrt{2} + 1\right)}\right| = 1$

Example, Phase Plot

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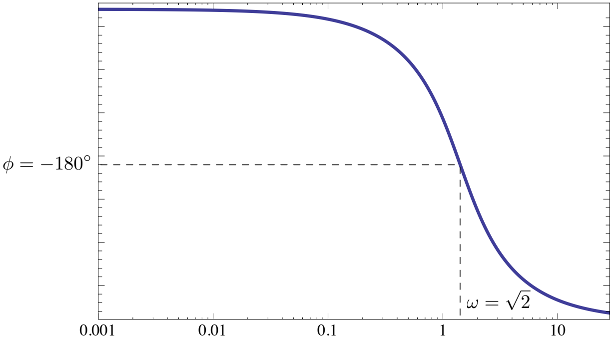
Phase plot (independent of K):



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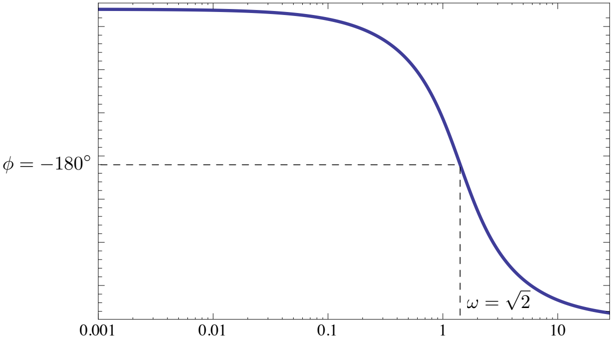
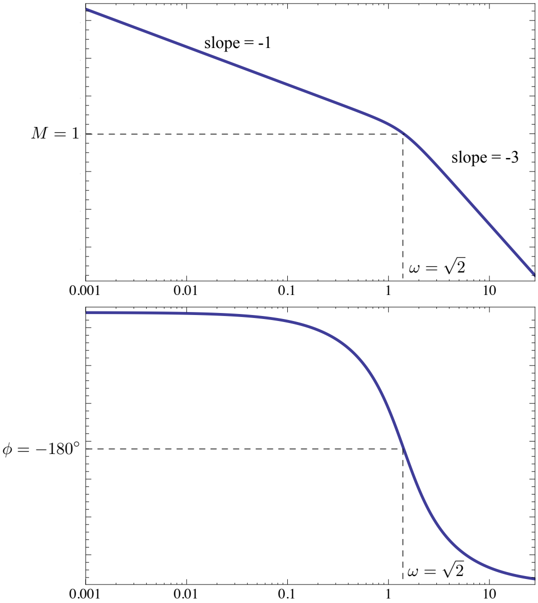


When $\omega = \sqrt{2}$, $\phi = -180°$

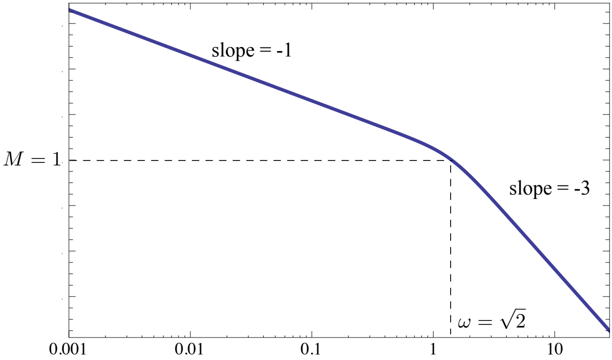


For the critical value $K = 4$:

$M = 1$ and $\phi = 180°$ mod $360°$ at $\omega = \sqrt{2}$

Crossover Frequency and Stability

Definition: The frequency at which $M = 1$ is called the *crossover frequency* and denoted by ω_c.



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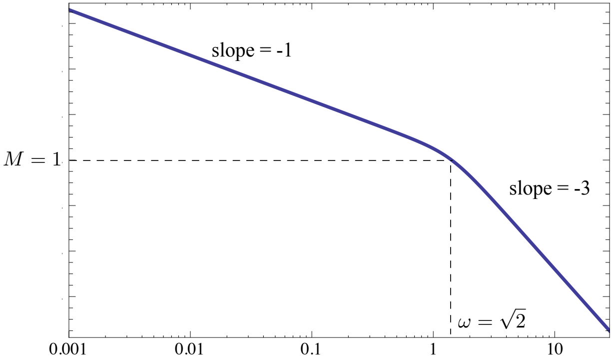
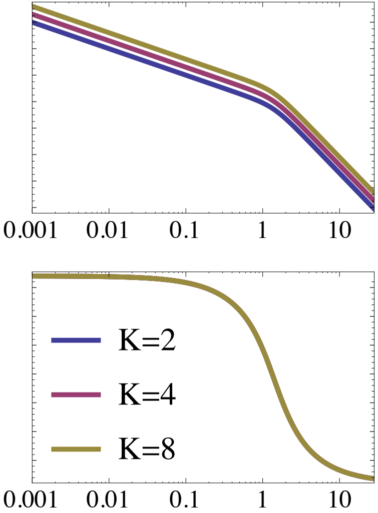


Transition from stability to instability on the Bode plot:

$$\text{for critical } K, \qquad \angle G(j\omega_c) = 180^\circ$$

Effect of Varying K



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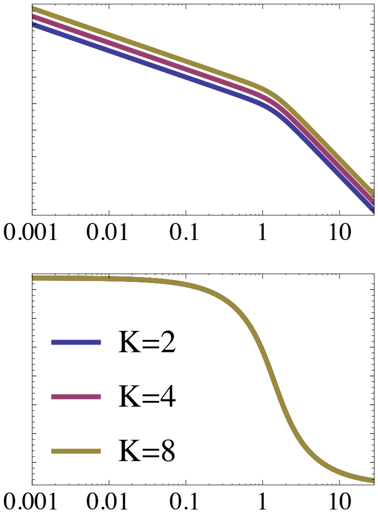

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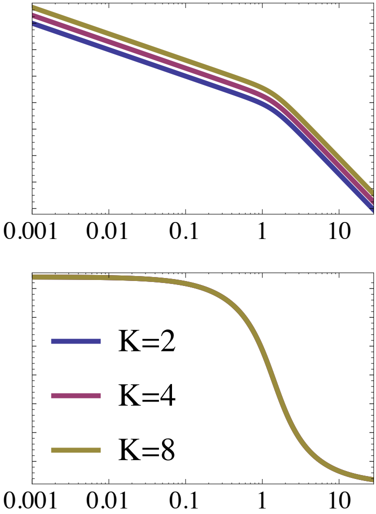

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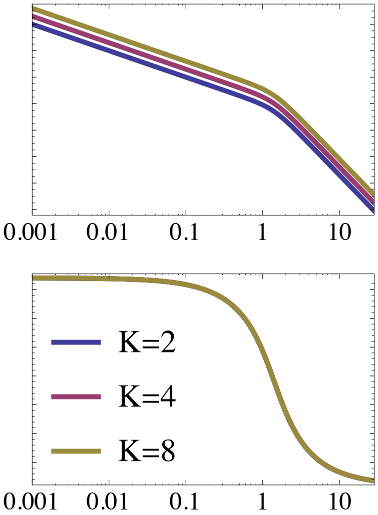

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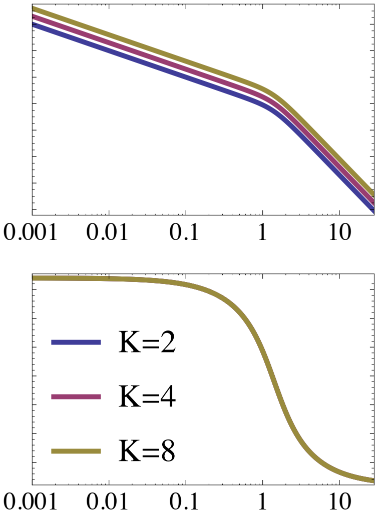

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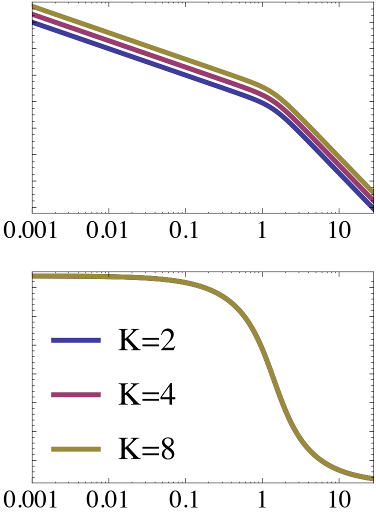

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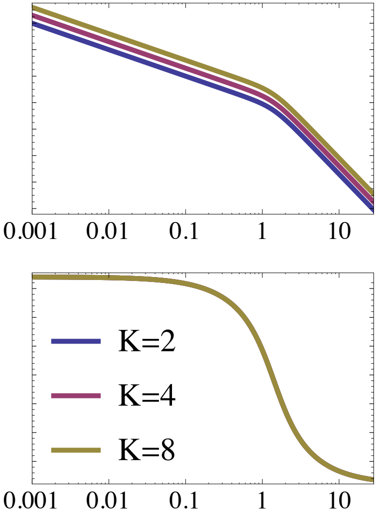

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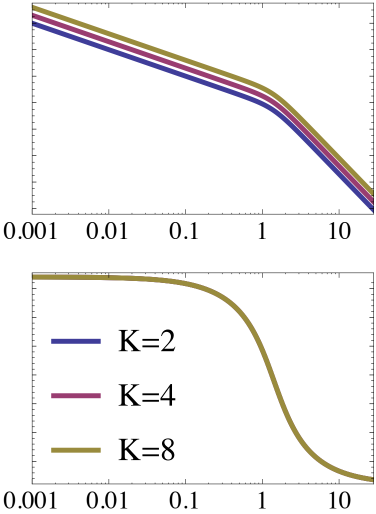

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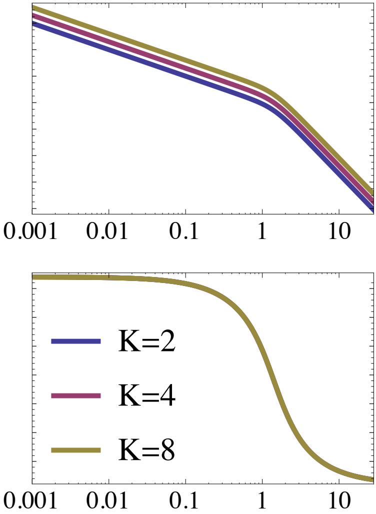

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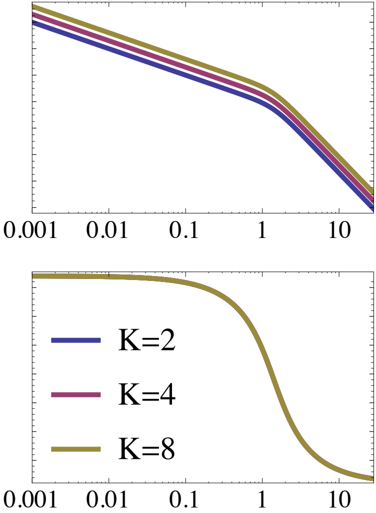

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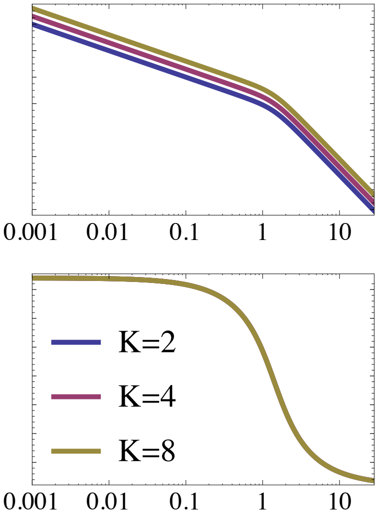

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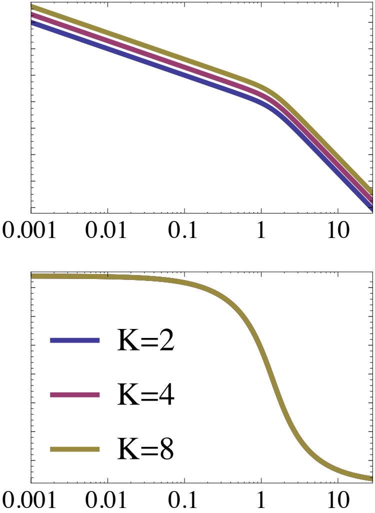

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$$\angle KG(j\omega_c) \begin{cases} > -180^\circ, & \text{for } K < 4 \\ & \text{(stable)} \\ = -180^\circ, & \text{for } K = 4 \\ & \text{(critical)} \\ < -180^\circ, & \text{for } K > 4 \\ & \text{(unstable)} \end{cases}$$

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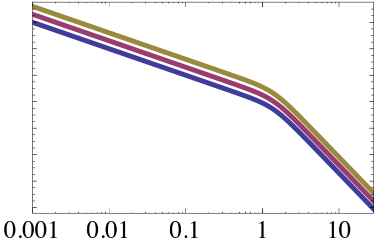

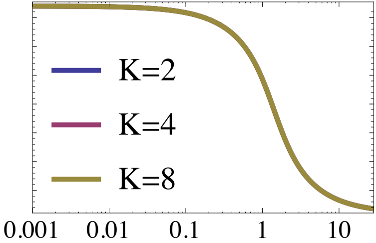

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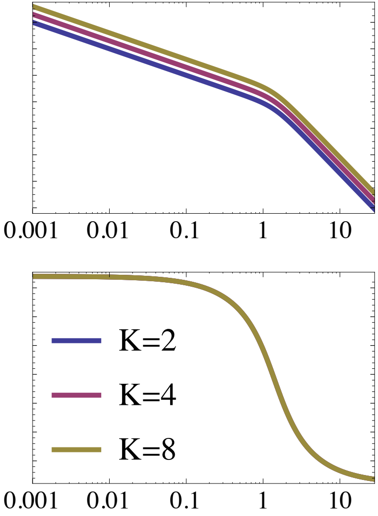

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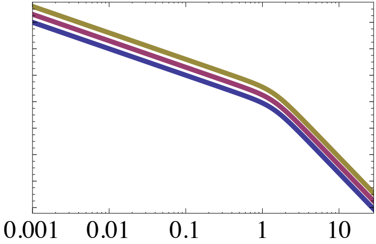

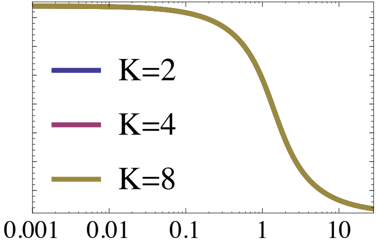

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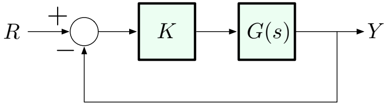
* Not a general rule; conditions will

vary depending on the system, must use either root locus or Nyquist plot to resolve ambiguity.

Stability from Frequency Response

Consider this unity feedback configuration:



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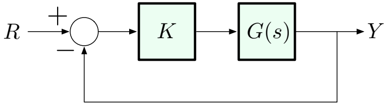

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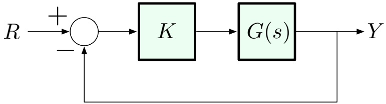

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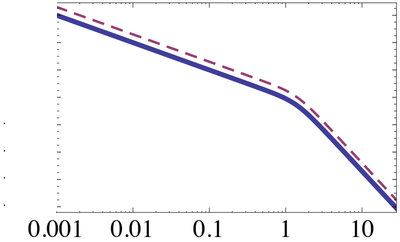
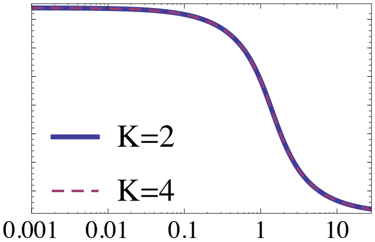
Two important characteristics: gain margin (GM) and phase margin (PM).

Gain Margin

Back to our example: $G(s) = \dfrac{1}{s(s^2+2s+2)}$, $K = 2$ (stable)





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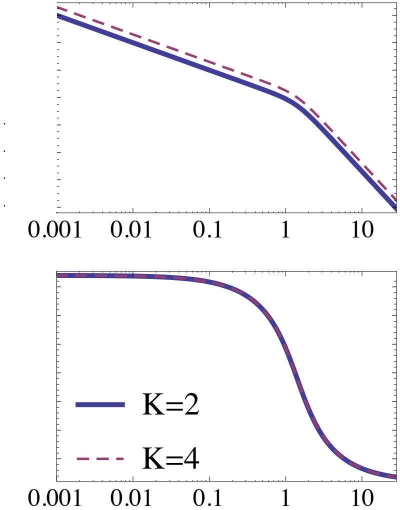

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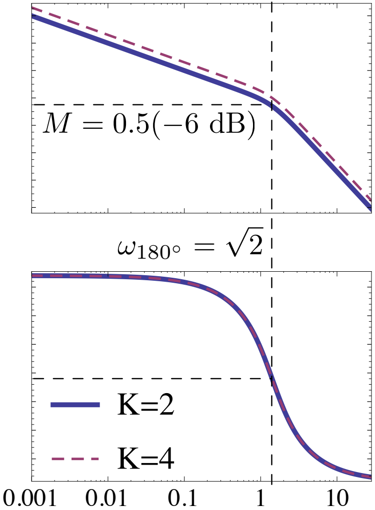

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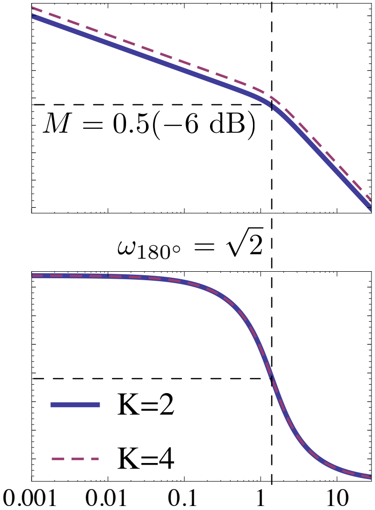

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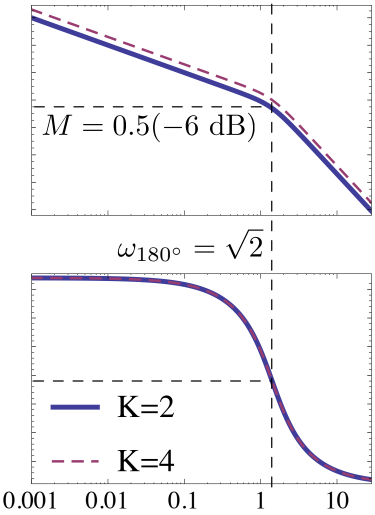

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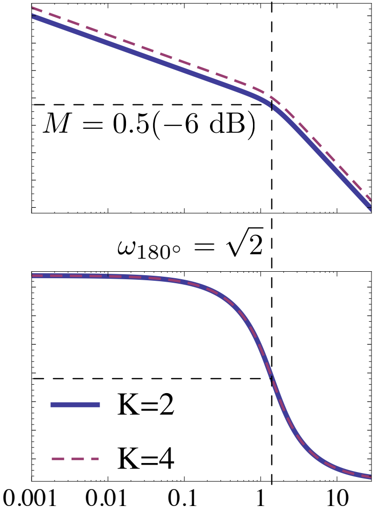

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In this example:

at $\omega_{180^\circ} = \sqrt{2}$

$M = 0.5\,(-6\text{ dB})$,

so GM $= 2$

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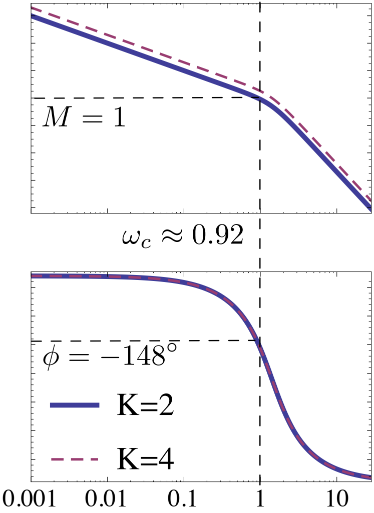

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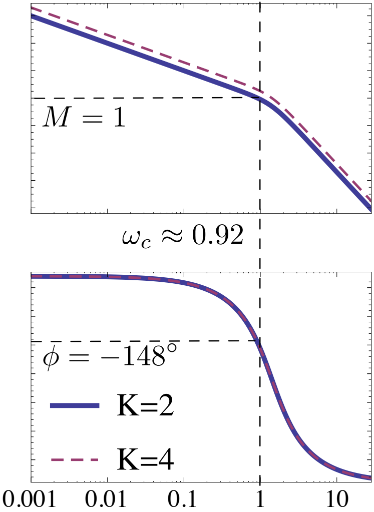

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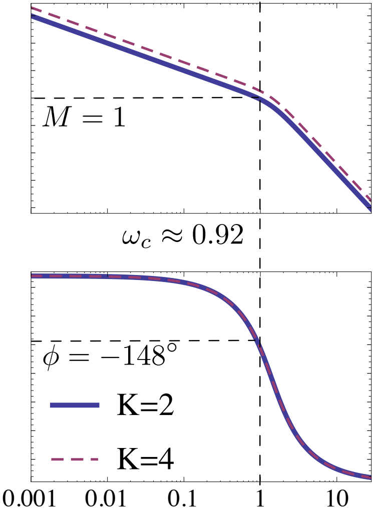

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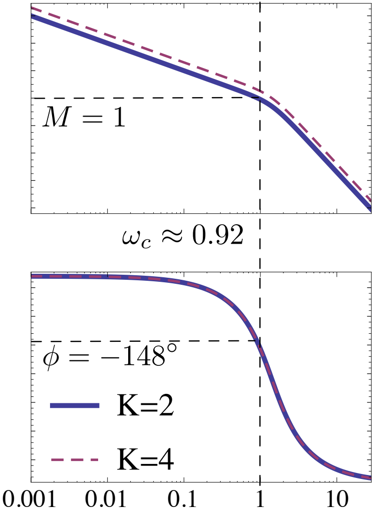

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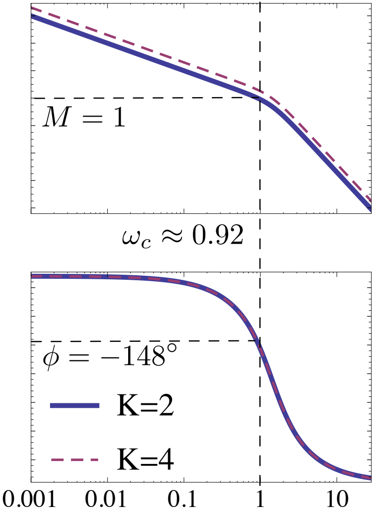

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(in practice, want $\text{PM} \geq 30^\circ$)

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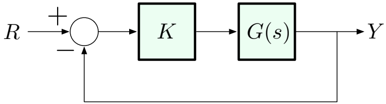

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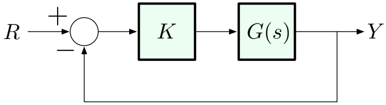

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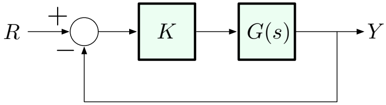

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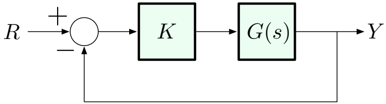

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Answer: $\mathrm{GM} = \infty$

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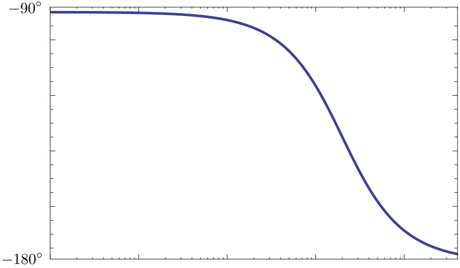

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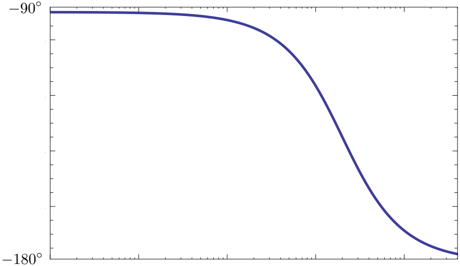

Recall: to find GM, we first need to find $\omega_{180°}$, and here there is no such $\omega \implies$ no GM.

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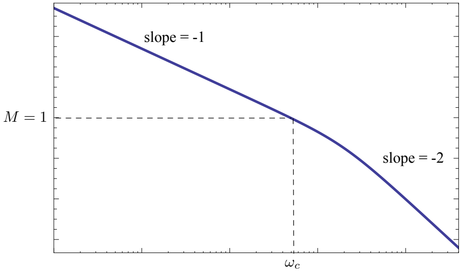

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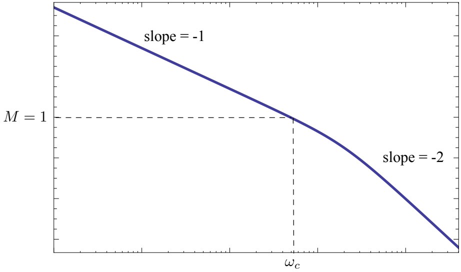

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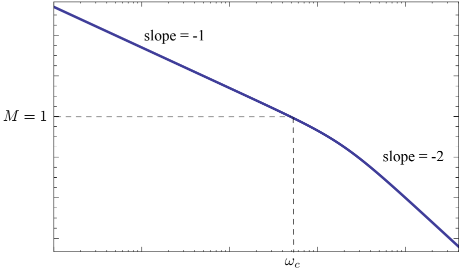

It can be shown that, *for this system*,

$$\mathrm{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{4\zeta^4 + 1} - 2\zeta^2}\right)$$

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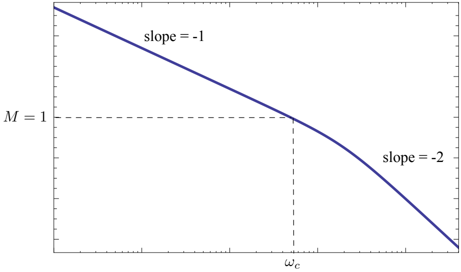

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— for PM $< 70°$, a good approximation is PM $\approx 100 \cdot \zeta$

Phase Margin for 2nd-Order System

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Conclusions:

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Thus, the overshoot $M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ and resonant peak $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$ are both related to PM through ζ!!

Preview: Bode's Gain-Phase Relationship

In the next lecture, we will see the following more generally:

Hendrik Wade Bode
(1905–1982)

Bode's Gain-Phase Relationship: all important characteristics of the closed-loop time response can be related to the phase margin of the open-loop transfer function!!

In fact, we will use a quantitative statement of this relationship as a design guideline.