ECE 486: Control Systems

Lecture 14B: Frequency Content of Signals

Key Takeaways

Bode plots can be used to gain intuition for how the system will respond to “low frequency” and “high frequency” signals.

The intuition follows from the following facts:

- The steady-state sinusoidal response for a stable system can be computed using the transfer function.
- By linear superposition, if the input is a sum of sinusoids then the steady state response is given by summing the responses due to each input sinusoid.
- General signals can be expressed as a sum of sinusoids using the Fourier Series. Signals can be roughly classified as low or high frequency based on the Fourier Series coefficients.

Low and High Frequency Signals

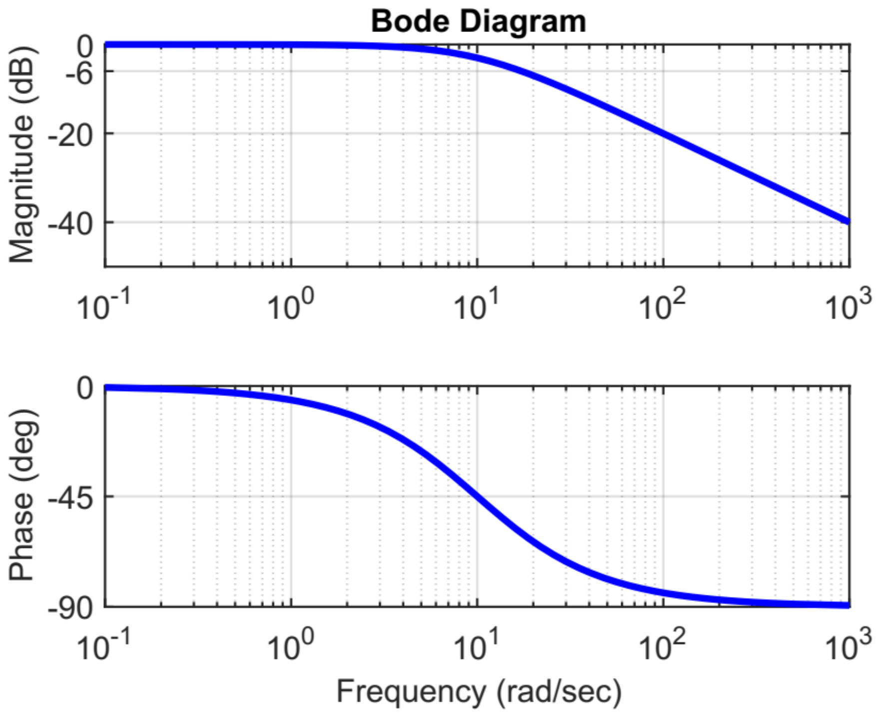
Consider the following stable, first-order system:

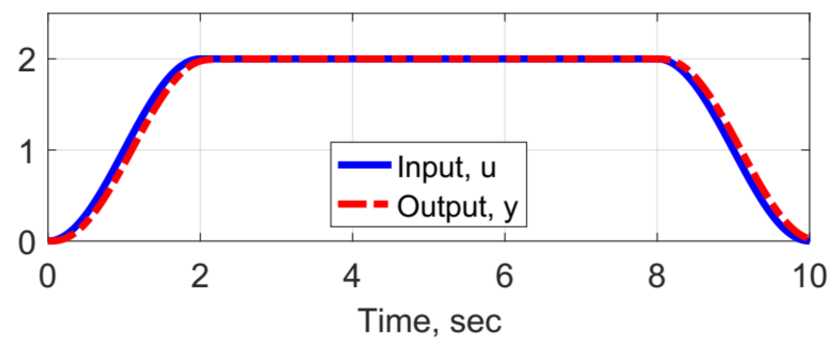
$$\dot{y}(t) + 10y(t) = 10u(t) \qquad G(s) = \frac{10}{s+10}$$

Steady-state frequency response with $u(t) = \cos(\omega t)$:

If $\omega \ll 10$ then $y(t) \approx u(t)$ in steady-state.

If $\omega \gg 10$ then $y(t) \approx 0$ in steady-state.





Low and High Frequency Signals

Consider the following stable, first-order system:

$$\dot{y}(t) + 10y(t) = 10u(t) \qquad G(s) = \frac{10}{s+10}$$

Steady-state frequency response with $u(t) = \cos(\omega t)$:

$$\text{If } \omega \ll 10 \text{ then } y(t) \approx u(t) \text{ in steady-state.}$$

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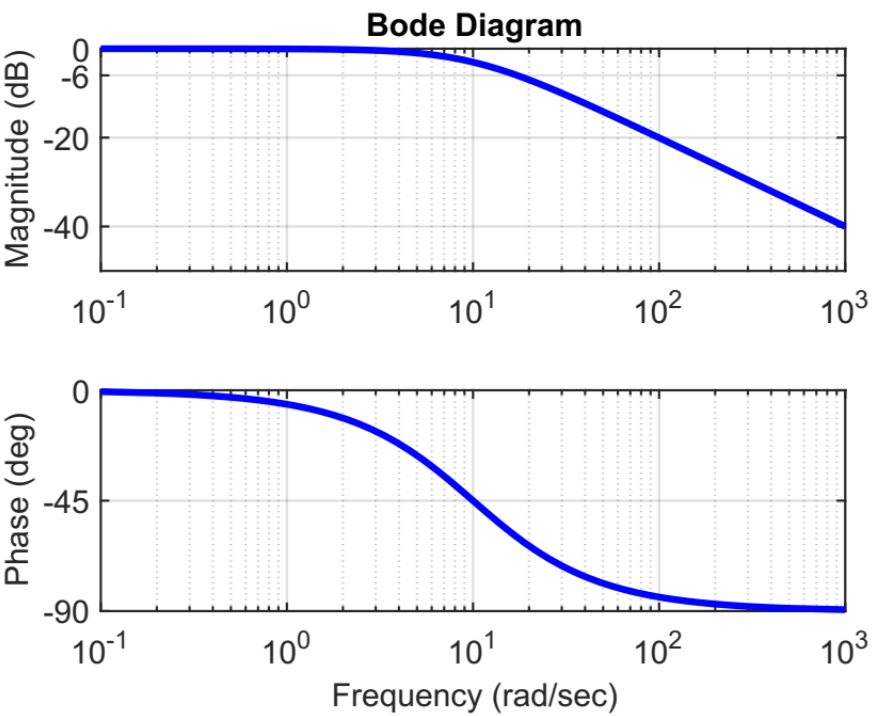
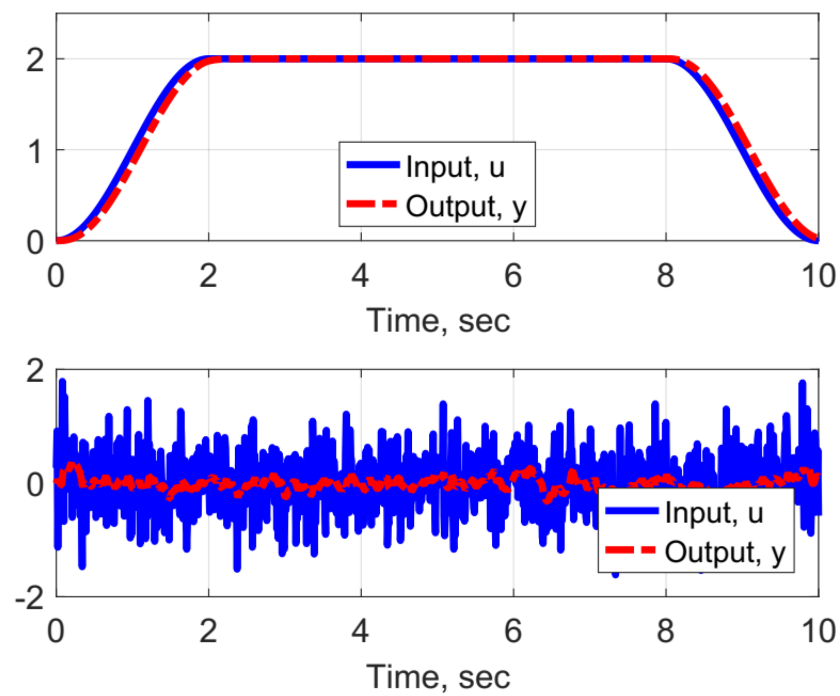
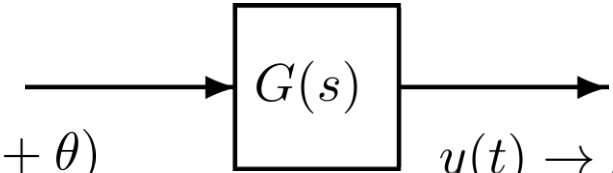




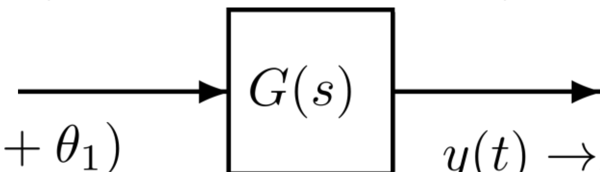
Superposition for Sinusoidal Inputs

The steady-state sinusoidal response for a stable, LTI system is:



$$u(t) = A\cos(\omega t + \theta) \qquad y(t) \to A|G(j\omega)|\cos\left(\omega t + \theta + \angle G(j\omega)\right)$$

By linear superposition, the response due to two sinusoids is:



$$u(t) = A_1\cos(\omega_1 t + \theta_1) + A_2\cos(\omega_2 t + \theta_2)$$

$$y(t) \to A_1|G(j\omega_1)|\cos\left(\omega_1 t + \theta_1 + \angle G(j\omega_1)\right) + A_2|G(j\omega_2)|\cos\left(\omega_2 t + \theta_2 + \angle G(j\omega_2)\right)$$

If the input is a sum of N sinusoids then the steady-state response is the sum of the response due to each input sinusoid.

Example

Consider the following stable, first-order system:

$$\dot{y}(t) + 10y(t) = 10u(t) \qquad G(s) = \frac{10}{s+10}$$

Force the system with the input:

$$u(t) = 1.2 + 0.9\cos\left(10t + \frac{\pi}{2}\right) + 0.9\cos\left(100t + \frac{\pi}{6}\right)$$

$G(0) = 1$

$$y(t) \to 1.2 +$$

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$G(10) = 0.5 - 0.5j = 0.707e^{-j\frac{\pi}{4}}$

$G(0) = 1$

$$y(t) \to 1.2 + 0.64\cos\left(10t + \frac{\pi}{4}\right) +$$

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$G(10) = 0.5 - 0.5j = 0.707e^{-j\frac{\pi}{4}}$

$G(0) = 1$

$G(100) = 0.01 - 0.1j = 0.1e^{-j1.47}$

$$y(t) \to 1.2 + 0.64\cos\left(10t + \frac{\pi}{4}\right) + 0.09\cos\left(100t - 0.95\right)$$

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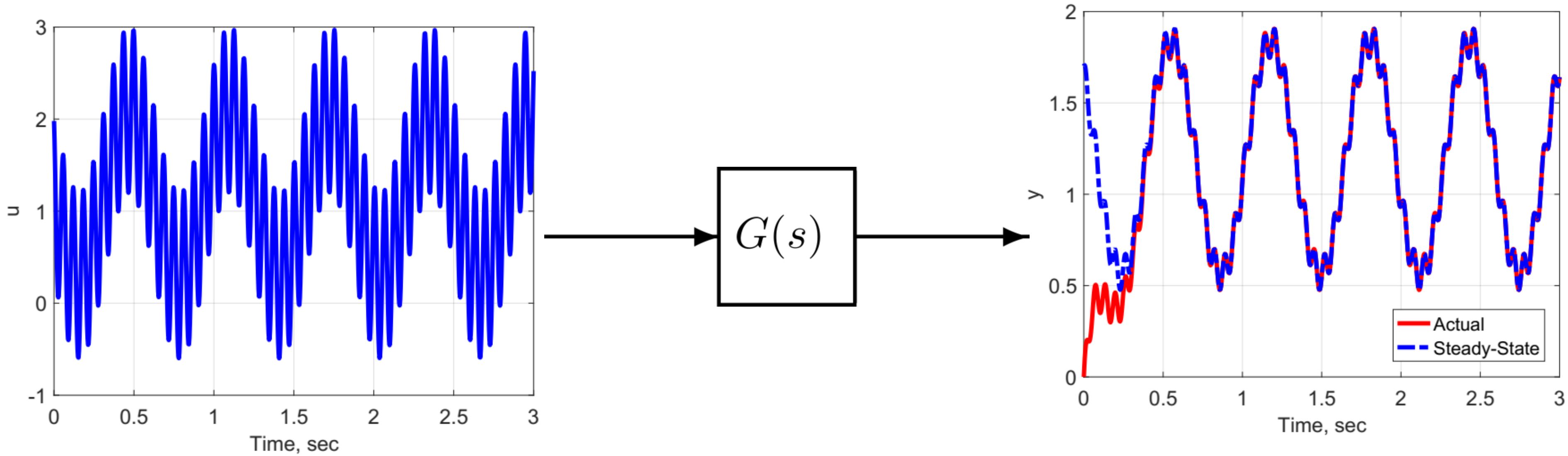


Fourier Series

A (real) input signal u(t) defined on [0,T] can be expressed as a sum of complex exponentials:

$$u(t) = \sum_{k=-\infty}^{\infty} U_k e^{j\omega_k t}$$

$$\text{where } U_k := \frac{1}{T}\int_0^T u(t) e^{-j\omega_k t} \text{ and } \omega_k := \frac{2\pi k}{T}$$

This can be re-written in terms of real sinusoids:

$$u(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \theta_k)$$

$$\text{where } A_k := |U_k| \text{ and } \theta_k := \angle U_k$$

Frequency Reponse

The Fourier Series, Principle of Superposition, and Bode plots can be used to understand how systems respond to signals:

1. Express u(t) defined on [0,T] as an infinite sum of cosine terms using the Fourier Series.

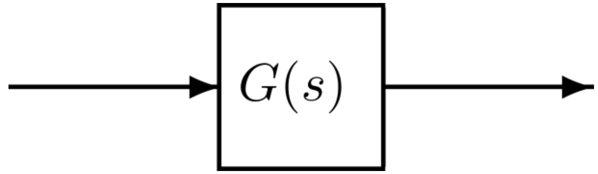
$$u(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \theta_k)$$

2. Compute the steady-state response due to the kth term.
3. By superposition, the steady-state response is:

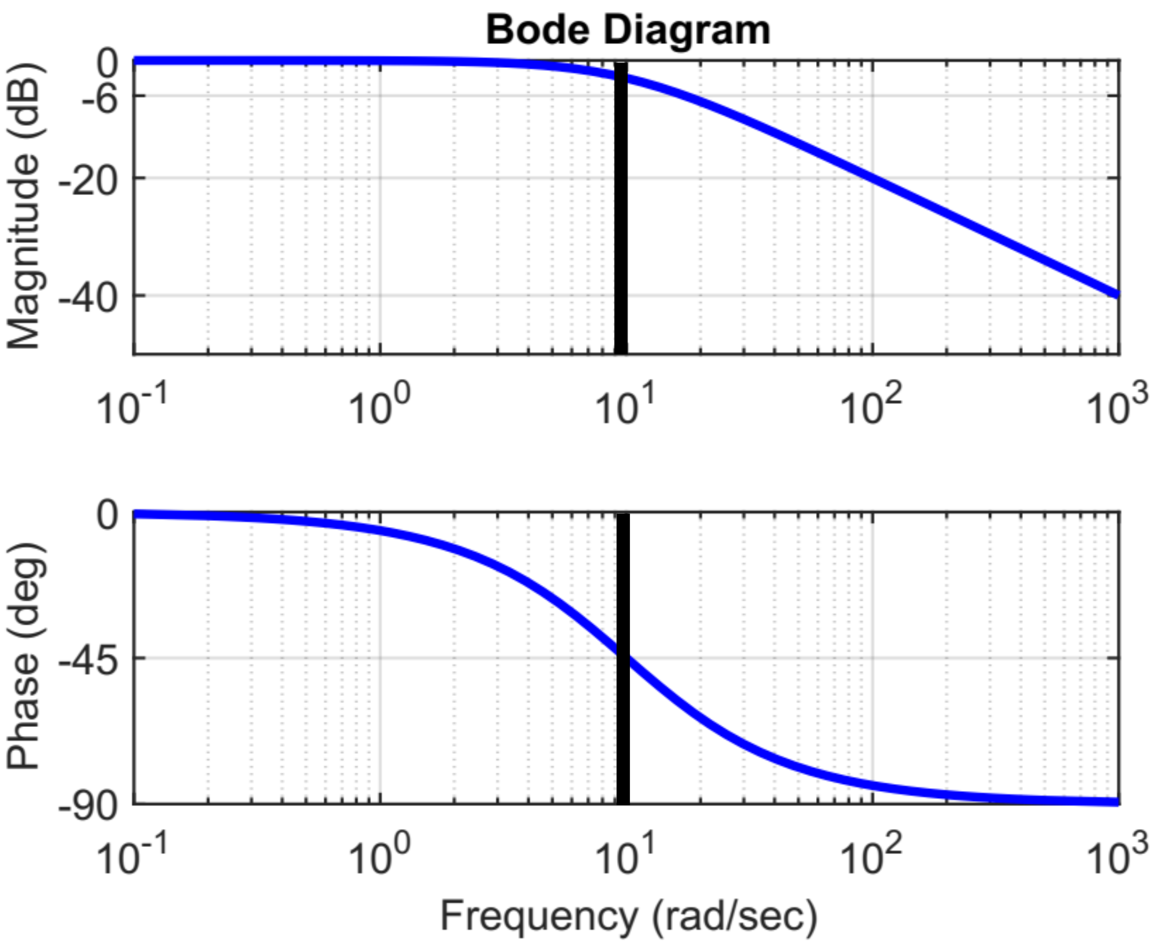
$$y(t) \to \sum_{k=0}^{\infty} A_k |G(j\omega_k)| \cos\left(\omega_k t + \theta_k + \angle G(j\omega_k)\right)$$

We won't use this formal procedure but it does motivate the use of informal terms "low" and "high" frequency signals.

Example



$$G(s) = \frac{10}{s+10}$$



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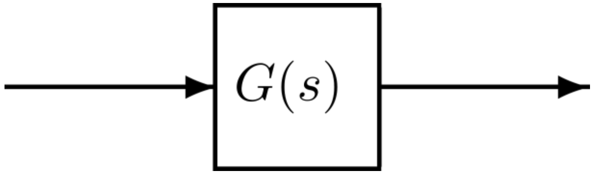

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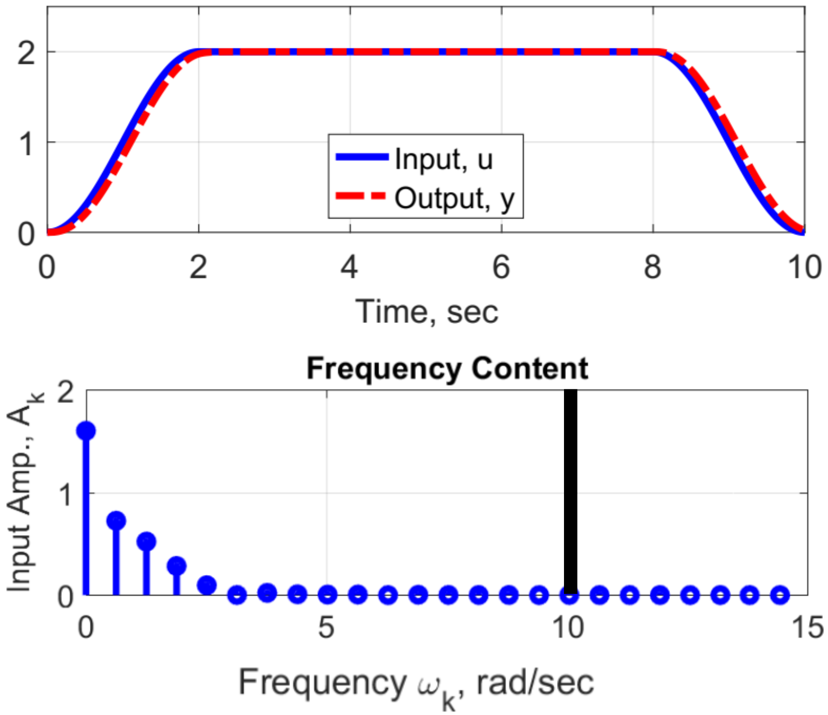

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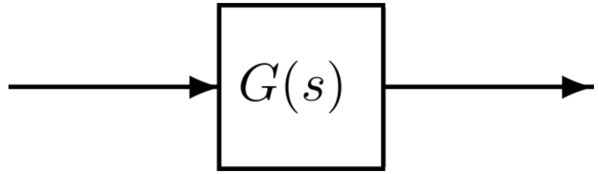

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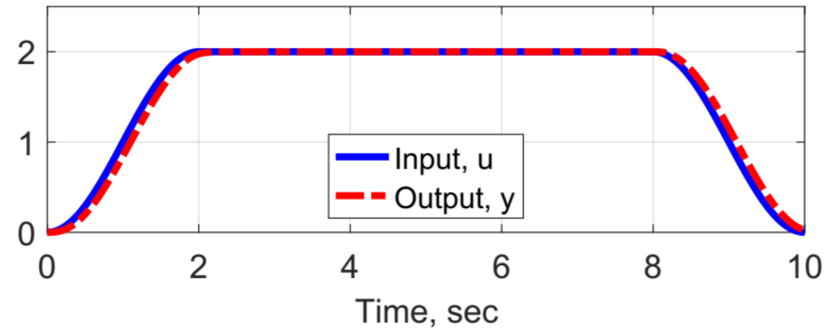

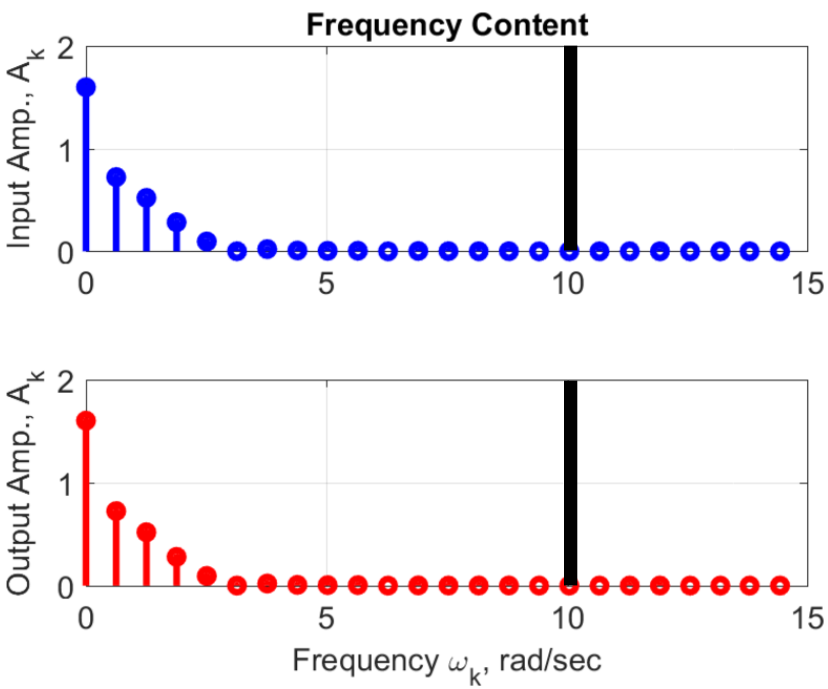

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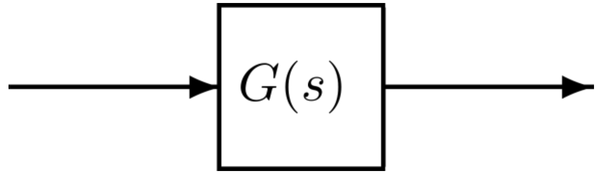

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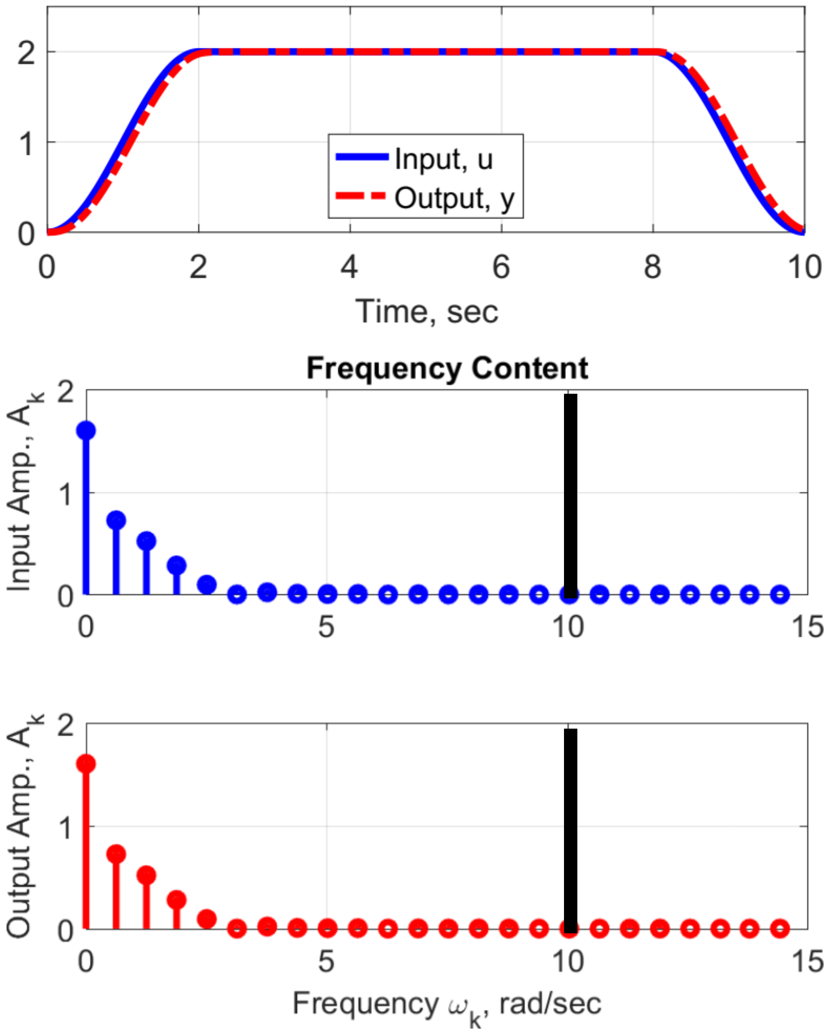

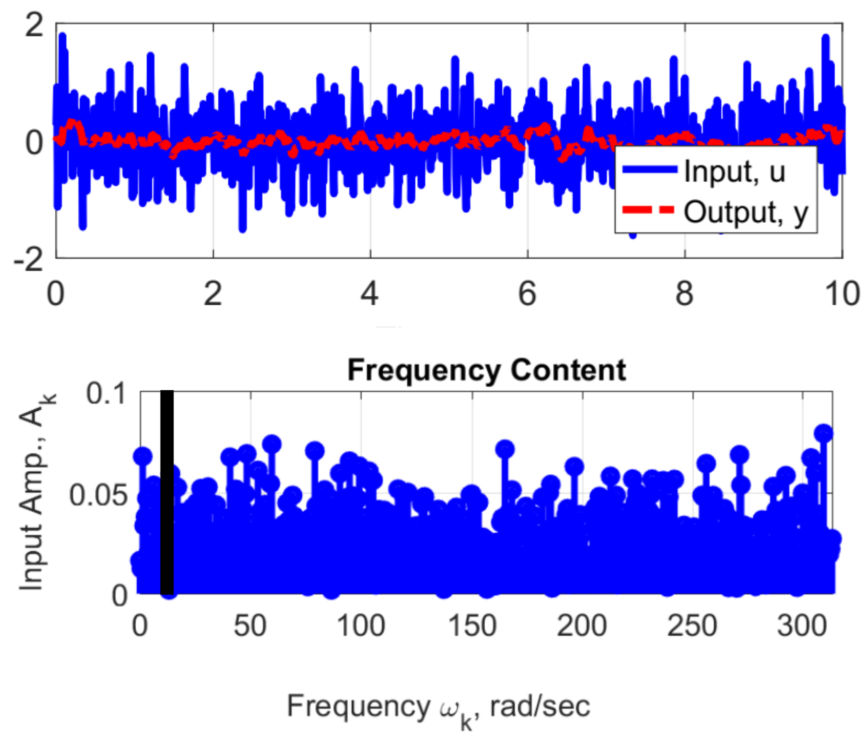

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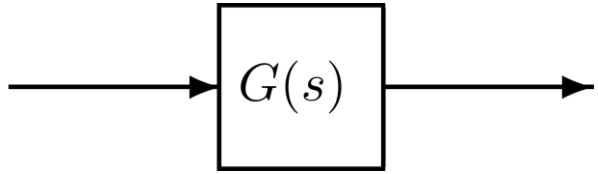

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