ECE 486: Control Systems

Lecture 18C: Nyquist Stability Condition

Key Takeaways

This lecture covers the Nyquist stability theorem.

Let L(s)=G(s)K(s) be the (open) loop transfer function.

- The value s=-1 is a critical point in the Nyquist plot.
- The closed-loop is unstable if the Nyquist curve $L(j\omega_0)=-1$ at some frequency ω_0.

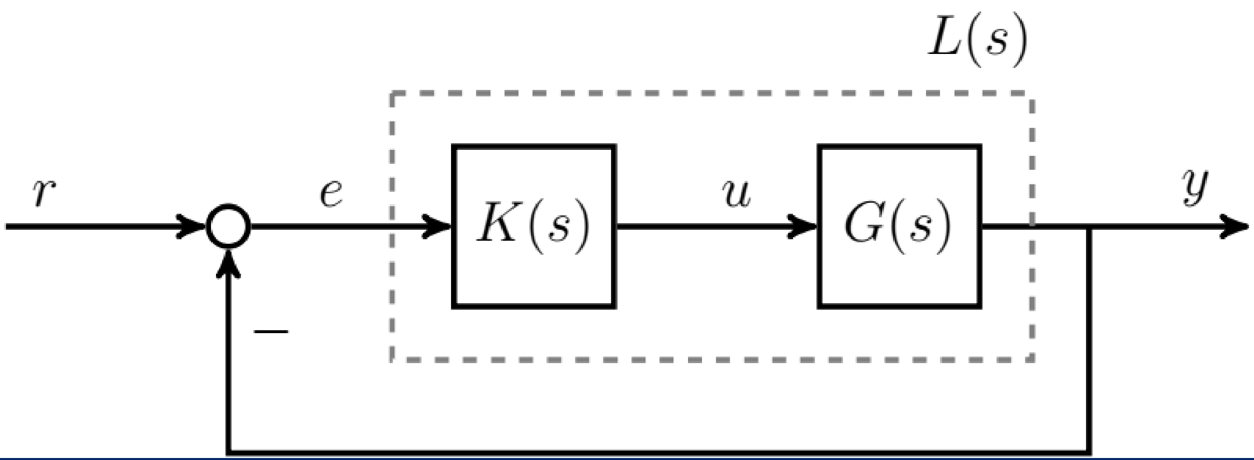
The Nyquist stability theorem states that the closed-loop is stable if and only if the Nyquist curve of *L(s)* encircles the $s = -1$ point the "correct" number of times. The "correct" number of times is equal to the number of RHP poles of the loop L(s).

Critical -1 Point

The transfer function *L(s) = G(s)K(s)* is called the (open) loop transfer function.

If the Nyquist curve of $L(s)$ passes through the critical point $s = -1$ then the closed-loop is unstable.

- Suppose $L(j\omega_0)=-1$ at some frequency ω_0. Hence $1+L(j\omega_0)=0$.
- The sensitivity $S(s) = \frac{1}{1+L(s)}$ has a pole on the imaginary axis at $s=j\omega_0$.



Example

```
>> G = tf(4,[1 2.0407 4]);
>> K = tf(20,[1 5]);
>> L = G*K;
>> nyquist(L);
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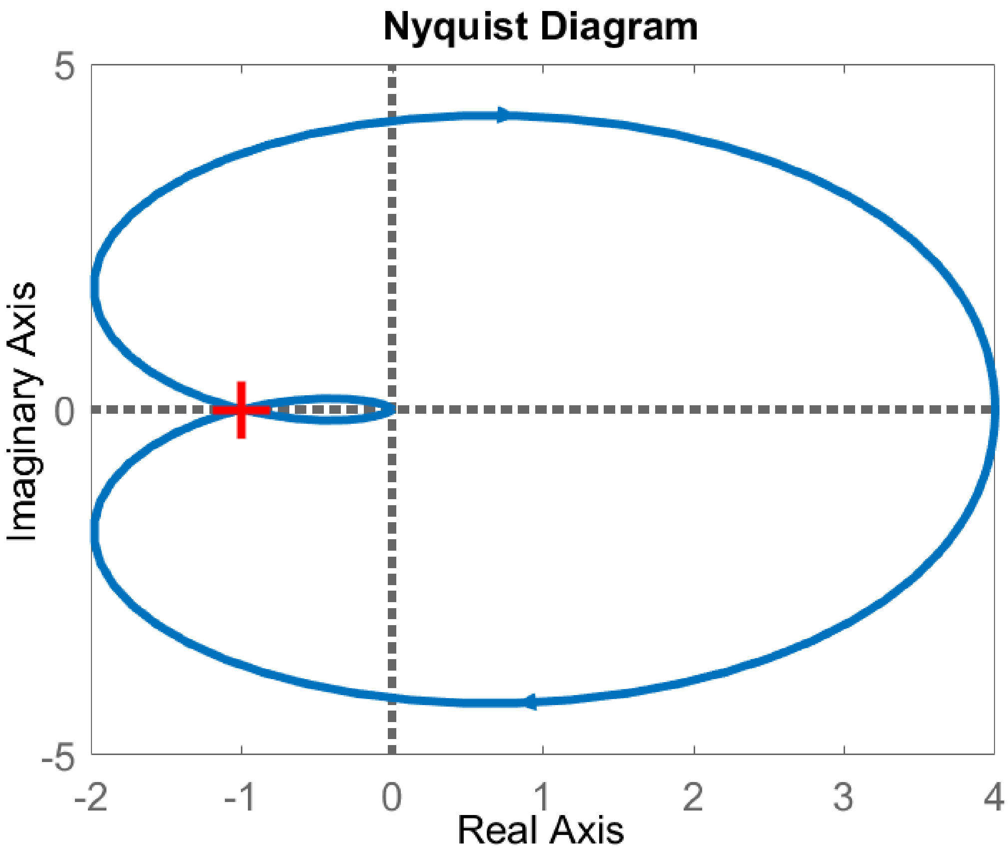
$$G(s) = \frac{4}{s^2+2.0407s+4}$$

$$K(s) = \frac{20}{s+5}$$

```
>> S=feedback(1,L);
>> pole(S)
ans =
  -7.0407 + 0.0000i
  -0.0000 + 3.7687i
  -0.0000 - 3.7687i
>> evalfr(L,1j*3.7687)
ans =
  -1.0000 - 0.0000i
```



Nyquist Theorem

Notation:

- P_{CL}: Number of poles of the closed-loop in the CRHP.
- P_{OL}: Number of poles of the open-loop $L(s)$ in the CRHP.
- N_{CCW}: This denotes the number of times the Nyquist curve of L(s) encircles the critical −1 point. $N_{CCW}>0$ for counterclockwise (CCW) encirclements and $N_{CCW}<0$ for clockwise (CW) encirclements.

Nyquist Theorem: Assume $L(s)=G(s)K(s)$ has no pole/zero cancellations in the CRHP and no poles on the imaginary axis. Then

$$P_{CL} = P_{OL} - N_{CCW.}$$

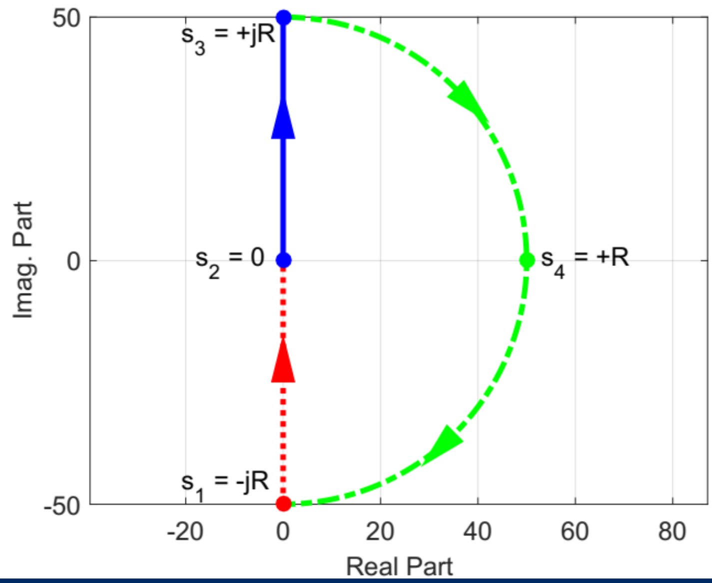
The closed-loop is stable ($P_{CL} = 0$) if and only if $N_{CCW} = P_{OL}$.

Benefit: Closed-loop stability can be determined from a Nyquist plot of the open loop transfer function L(s).

Nyquist Theorem

The Nyquist theorem follows from Cauchy's Argument Principle.

- Consider the curve Γ given by Γ_R as $R \rightarrow \infty$. This encloses the RHP and L(Γ) is the Nyquist plot of L(s).
- Define *H(s)=1+L(s)*. H(Γ) encircles the origin N_z-N_p times CW.
- The Nyquist plot L(Γ) encircles the -1 point $N_{CCW}=N_p-N_z$ times CCW.
- RHP zeros of H(s) are the RHP poles of closed-loop: $N_z=P_{CL}$.
- RHP poles of H(s) are the RHP poles of L(s): $N_p = P_{OL}$.



Combining these facts:

$$P_{CL} = P_{OL} - N_{CCW.}$$

The theorem can be extended if L(s) has a pole on the imaginary axis.

Example 1

Loop $L_1(s) = \frac{2}{s+4}$

- $P_{OL} = 0$
- $N_{CCW} = 0$

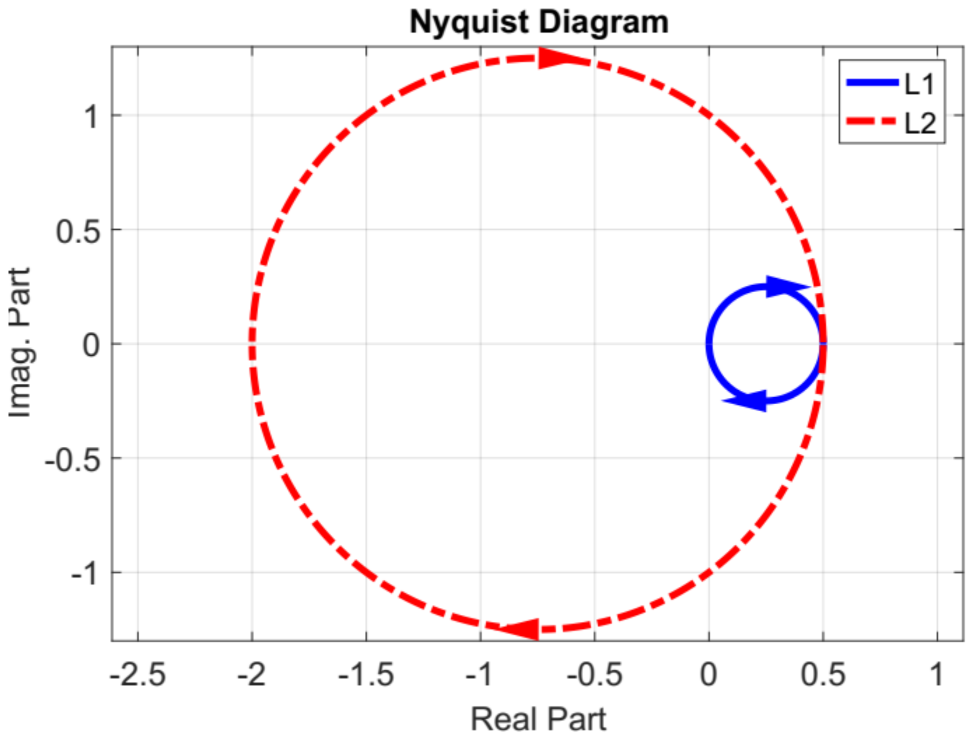
$\rightarrow\ P_{CL} = P_{OL} - N_{CCW.} = 0.$

Closed-loop is stable.

Verify:

$$S_1(s) = \frac{1}{1+L_1(s)}$$

$$= \frac{1}{1+\frac{2}{s+4}} = \frac{s+4}{s+6}$$



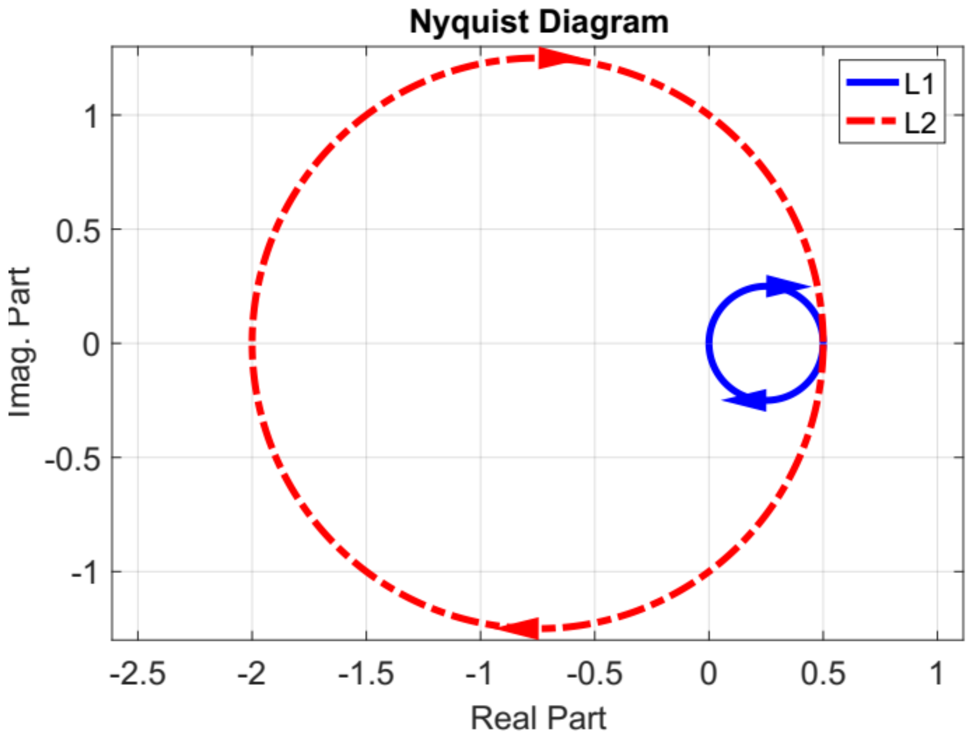
Example 2

Loop $L_2(s) = \frac{-2s+2}{s+4}$

- $P_{OL} = 0$
- $N_{CCW} = -1$

$\rightarrow \; P_{CL} = P_{OL} - N_{CCW.} = +1.$

Closed-loop is unstable.



Verify:

$$S_2(s) = \frac{1}{1 + L_2(s)}$$

$$= \frac{1}{1 + \frac{-2s+2}{s+4}} = \frac{s+4}{-s+6}$$

Example 3

Loop $L_3(s) = \frac{2}{s-4}$

- $P_{OL} = 1$
- $N_{CCW} = 0$

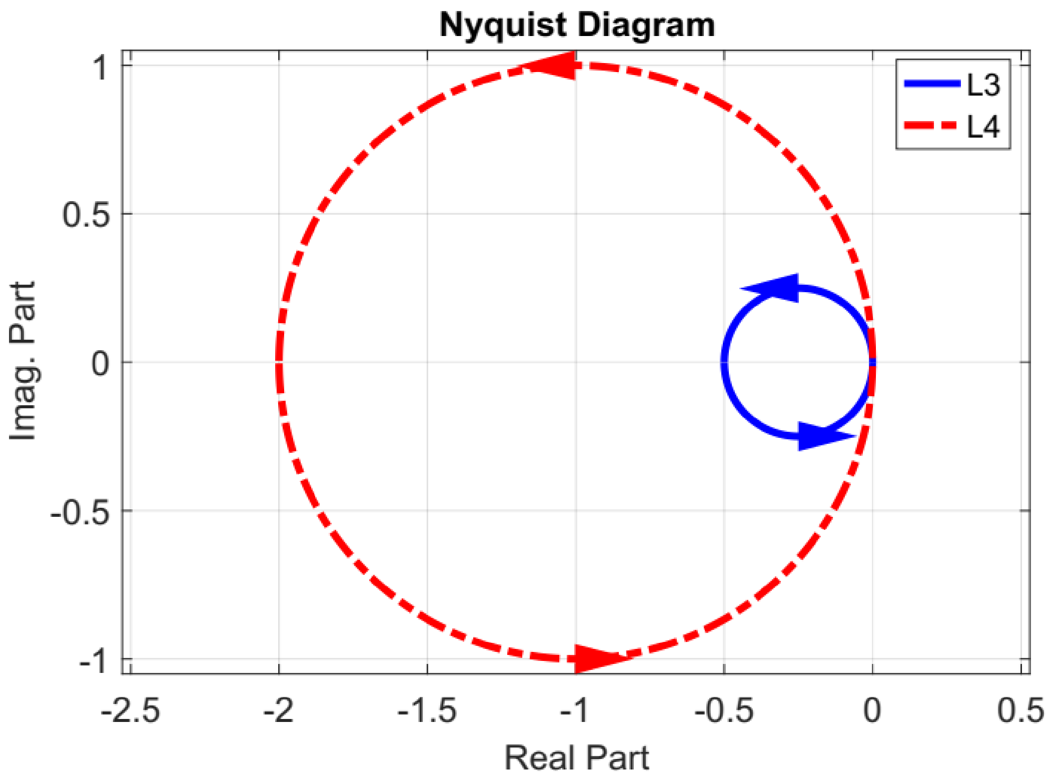
$\rightarrow$ $P_{CL} = P_{OL} - N_{CCW.} = +1.$

Closed-loop is unstable.

Verify:

$$S_3(s) = \frac{1}{1 + L_3(s)}$$

$$= \frac{1}{1 + \frac{2}{s-4}} = \frac{s-4}{s-2}$$



Example 4

Loop $L_4(s) = \frac{8}{s-4}$

- $P_{OL} = 1$
- $N_{CCW} = 1$

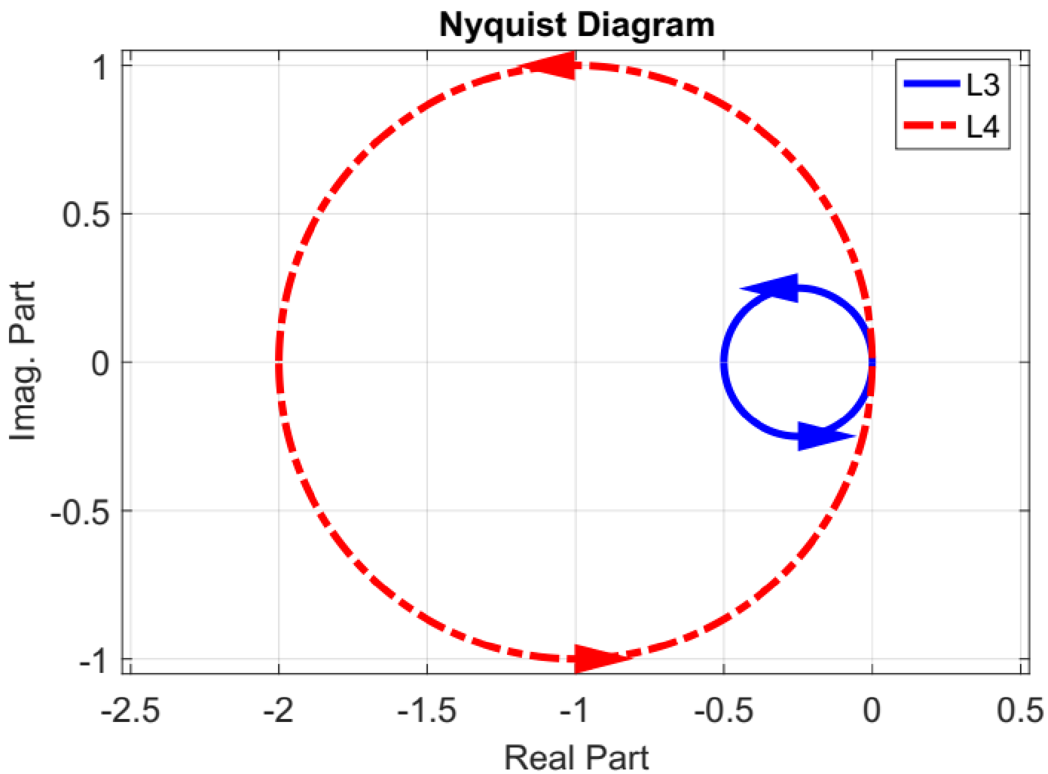
$\rightarrow$ $P_{CL} = P_{OL} - N_{CCW.} = 0.$

Closed-loop is stable.

Verify:

$$S_4(s) = \frac{1}{1 + L_4(s)}$$

$$= \frac{1}{1 + \frac{8}{s-4}} = \frac{s-4}{s+4}$$



Example 5

Loop $L_5(s) = \frac{2}{s-5}\frac{100}{s^2+5s+100}$

- $P_{OL} = 1$
- $N_{CCW} = 1$

$\rightarrow$ $P_{CL} = P_{OL} - N_{CCW.} = 0.$

Closed-loop is stable.