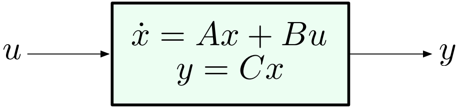
ECE 486: Control Systems

- Lecture 22: controllability, stability, and pole-zero cancellations; effect of coordinate transformations; conversion of any controllable system to CCF.

Goal: explore the effect of pole-zero cancellations on internal stability; understand the effect of coordinate transformations on the properties of a given state-space model (transfer function; open-loop poles; controllability).

Reading: FPE, Chapter 7

State-Space Realizations



- a given transfer function $G(s)$ can be realized using infinitely many state-space models
- certain properties make some realizations preferable to others
- one such property is *controllability*

Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A, B) = \left[B \,|\, AB \,|\, A^2 B \,|\, \ldots \,|\, A^{n-1} B \right]$$

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

- As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.

Example: Computing $\mathcal{C}(A, B)$

Let's get back to our old friend:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2 \times 2} \implies \mathcal{C}(A, B) \in \mathbb{R}^{2 \times 2}$

$$\mathcal{C}(A, B) = [B \,|\, AB] \qquad AB = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$$\implies \mathcal{C}(A, B) = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}$$

Is this system controllable?

$$\det \mathcal{C} = -1 \neq 0 \qquad \implies \qquad \text{system is controllable}$$

Controller Canonical Form

A single-input state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Controller Canonical Form (CCF) is the matrices A, B are of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A system in CCF is *always controllable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

CCF with Arbitrary Zeros

In our example, we had $G(s) = \dfrac{s+1}{s^2+5s+6}$, with a minimum-phase zero at $z = -1$.

Let's consider a general zero location $s = z$:

$$G(s) = \frac{s-z}{s^2+5s+6}$$

This gives us a CCF realization

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} -z & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since A, B are the same, $\mathcal{C}(A, B)$ is the same $\Longrightarrow$ the system is still controllable.

A system in CCF is controllable for any locations of the zeros.

OCF with Arbitrary Zeros

Start with the CCF

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} -z & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Convert to OCF: $\qquad (A \mapsto A^T, B \mapsto C^T, C \mapsto B^T)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}}_{\bar{A}=A^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} -z \\ 1 \end{pmatrix}}_{\bar{B}=C^T} u, \qquad y = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{\bar{C}=B^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We already know that this system realizes the same t.f. as the original system.

But is it *controllable*?

OCF with Arbitrary Zeros

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}}_{\bar{A}=A^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} -z \\ 1 \end{pmatrix}}_{\bar{B}=C^T} u, \qquad y = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{\bar{C}=B^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let's find the controllability matrix:

$$\mathcal{C}(\bar{A}, \bar{B}) = \left[\bar{B} \,|\, \bar{A}\bar{B} \right] \qquad \bar{A}\bar{B} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} -z \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -z-5 \end{pmatrix}$$

$$\therefore \mathcal{C}(\bar{A}, \bar{B}) = \begin{pmatrix} -z & -6 \\ 1 & -z-5 \end{pmatrix}$$

$$\det \mathcal{C} = z(z+5) + 6 = z^2 + 5z + 6 = 0 \quad \text{for } z = -2 \text{ or } z = -3$$

The OCF realization of the transfer function $G(s) = \dfrac{s-z}{s^2+5s+6}$ is not controllable when $z = -2$ or -3, even though the CCF is always controllable.

Beware of Pole-Zero Cancellations!

The OCF realization of the transfer function

$$G(s) = \frac{s-z}{s^2+5s+6}$$

is not controllable when $z = -2$ or -3, even though the CCF is always controllable.

Let's examine $G(s)$ when $z = -2$:

$$G(s) = \frac{s-z}{s^2+5s+6}\bigg|_{z=-2} = \frac{\cancel{s+2}}{(\cancel{s+2})(s+3)} = \frac{1}{s+3}$$

— pole-zero cancellation!

For $z = -2$, $G(s)$ is a first-order transfer function, which can always be realized by this 1st-order controllable model:

$$\dot{x}_1 = -3x_1 + u, \; y = x_1 \quad \longrightarrow \quad G(s) = \frac{1}{s+3}$$

Beware of Pole-Zero Cancellations!!

We can look at this from another angle: consider the t.f.

$$G(s) = \frac{1}{s+3}$$

We can realize it using a one-dimensional controllable state-space model

$$\dot{x}_1 = -3x_1 + u, \quad y = x_1$$

or a noncontrollable two-dimensional state-space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

— certainly not the best way to realize a simple t.f.!

Thus, even the *state dimension* of a realization of a given t.f. is not unique!!

Beware of Pole-Zero Cancellations!!

Here is a really bad realization of the t.f.

$$G(s) = \frac{1}{s+3}.$$

Use a two-dimensional model:

$$\begin{aligned} \dot{x}_1 &= -3x_1 + u \\ \dot{x}_2 &= 100x_2 \\ y &= x_1 \end{aligned}$$

- x_2 is not affected by the input u (i.e., it is an uncontrollable mode), and not visible from the output y
- does not change the transfer function
- ... and yet, horrible to implement: $x_2(t) \propto e^{100t}$

The transfer function can mask undesirable internal state behavior!!

Pole-Zero Cancellations and Stability

- In case of a pole-zero cancellation, the t.f. contains *much less* information than the state-space model because some dynamics are “hidden.”
- These dynamics can be either good (stable) or bad (unstable), but we cannot tell from the t.f.
- Our original definition of stability (no RHP poles) is flawed because there can be RHP eigenvalues of the system matrix A that are canceled by zeros,yet they still have dynamics associated with them.

Definition of Internal Stability (State-Space Version): a state-space model with matrices (A, B, C, D) is *internally stable* if all eigenvalues of the A matrix are in LHP.

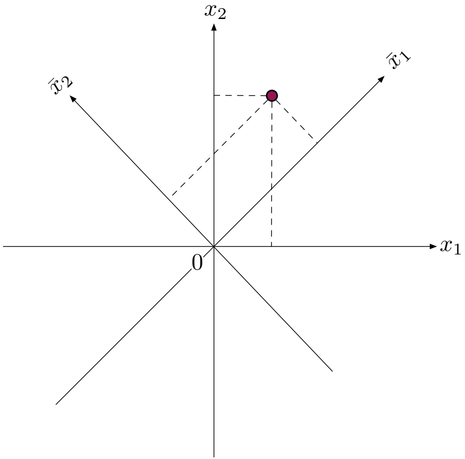
This is equivalent to having no RHP open-loop poles and no pole-zero cancellations in RHP.

Coordinate Transformations

Now that we have seen that a given transfer function can have many different state-space realizations, we would like a systematic procedure of generating such realizations, preferably with favorable properties (like controllability).

One such procedure is by means of *coordinate transformations*.

Coordinate Transformations



$x \longmapsto \bar{x} = Tx,$ $\qquad T \in \mathbb{R}^{n \times n}$nonsingular

$x = T^{-1}\bar{x}$ $\qquad$ (go back and forth between the coordinate systems)

Coordinate Transformations

For example,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

This can be represented as

$$\bar{x} = Tx, \qquad \text{where } T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The transformation is invertible: $\det T = -2$, and

$$T^{-1} = \frac{1}{\det T} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Or we can see this directly:

$$\bar{x}_1 + \bar{x}_2 = 2x_1; \quad \bar{x}_1 - \bar{x}_2 = 2x_2$$

Coordinate Transformations and State-Space Models

Consider a state-space model

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

and a change of coordinates $\bar{x} = Tx$ (T invertible).

What does the system look like in the new coordinates?

$$\begin{aligned}\dot{\bar{x}} &= \dot{\overline{Tx}} = T\dot{x} && \text{(linearity of derivative)} \\ &= T(Ax + Bu) \\ &= T(AT^{-1}\bar{x} + Bu) && (x = T^{-1}\bar{x}) \\ &= \underbrace{TAT^{-1}}_{\bar{A}}\bar{x} + \underbrace{TB}_{\bar{B}}u \\ y &= Cx \\ &= \underbrace{CT^{-1}}_{\bar{C}}\bar{x}\end{aligned}$$

Coordinate Transformations and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \xrightarrow{\quad T \quad} \qquad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where

$$\bar{A} = TAT^{-1}, \qquad \bar{B} = TB, \qquad \bar{C} = CT^{-1}$$

What happens to

- the transfer function?
- the controllability matrix?

Coordinate Transformations and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \xrightarrow{\quad T \quad} \qquad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}, \qquad \bar{B} = TB, \qquad \bar{C} = CT^{-1}$

Claim: The transfer function doesn't change.

Proof:

$$\begin{aligned} \bar{G}(s) &= \bar{C}(Is - \bar{A})^{-1}\bar{B} \\ &= (CT^{-1})\left(Is - TAT^{-1}\right)^{-1}(TB) \\ &= CT^{-1}\left(TIT^{-1}s - TAT^{-1}\right)^{-1}TB \\ &= CT^{-1}\left[T\left(Is - A\right)T^{-1}\right]^{-1}TB \\ &= C\underbrace{T^{-1}T}_{I}(Is - A)^{-1}\underbrace{T^{-1}T}_{I}B \\ &= C\left(Is - A\right)^{-1}B \equiv G(s) \end{aligned}$$

Coordinate Transformations and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{\quad T \quad} \quad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}$, $\qquad \bar{B} = TB$, $\qquad \bar{C} = CT^{-1}$

The transfer function doesn't change.

In fact:

- open-loop poles don't change
- characteristic polynomial doesn't change:

$$\begin{aligned} \det(Is - \bar{A}) &= \det(Is - TAT^{-1}) \\ &= \det\left[T(Is - A)T^{-1}\right] \\ &= \det T \cdot \det(Is - A) \cdot \det T^{-1} \\ &= \det(Is - A) \end{aligned}$$

Coordinate Transformations and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{T} \quad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}$, $\qquad \bar{B} = TB$, $\qquad \bar{C} = CT^{-1}$

Claim: Controllability doesn't change.

Proof: For any $k = 0, 1, \ldots$,

$$\bar{A}^k \bar{B} = (TAT^{-1})^k TB = TA^k T^{-1} TB = TA^k B \qquad \text{(by induction)}$$

Therefore,
$$\begin{aligned} \mathcal{C}(\bar{A}, \bar{B}) &= [TB \,|\, TAB \,|\, \ldots \,|\, TA^{n-1}B] \\ &= T[B \,|\, AB \,|\, \ldots \,|\, A^{n-1}B] \\ &= T\mathcal{C}(A, B) \end{aligned}$$

Since $\det T \neq 0$, $\det \mathcal{C}(\bar{A}, \bar{B}) \neq 0$ if and only if $\det \mathcal{C}(A, B) \neq 0$.

Thus, the new system is controllable if and only if the old one is.

Coordinate Transformations and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{\quad T \quad} \quad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}, \qquad \bar{B} = TB, \qquad \bar{C} = CT^{-1}$

Note: The *controllability matrix* does change:

$$\underbrace{\mathcal{C}(\bar{A},\bar{B})}_{\text{new}} = \underbrace{T}_{\substack{\text{coord.}\\ \text{trans.}}} \underbrace{\mathcal{C}(A,B)}_{\text{old}}$$

$$\Updownarrow$$

$$T = \mathcal{C}(\bar{A},\bar{B})\,[\mathcal{C}(A,B)]^{-1}$$

This is a recipe for going from one *controllable* realization of a given t.f. to another.

CCF is the most convenient controllable realization of a given t.f., so we want to *convert a given controllable system to CCF* (useful for control design).

Example: Converting a Controllable System to CCF

Note!! The way I do this is different from the textbook.

Consider $A = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (C is immaterial).

Convert to CCF if possible.

Step 1: check for controllability.

$$AB = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -8 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{C} = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}$$

$\det \mathcal{C} = -1$ – controllable

Example: Converting a Controllable System to CCF

Step 2: Determine desired $\mathcal{C}(\bar{A}, \bar{B})$.

We need to figure out $\bar{A}$ and $\bar{B}$.

For CCF, we must have

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \qquad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so we need to find the coefficients a_1, a_2.

Recall: the characteristic polynomial does not change:

$$\begin{aligned} \det(Is - A) &= \det(Is - \bar{A}) \\ \det \begin{pmatrix} s+15 & -8 \\ 15 & s-7 \end{pmatrix} &= \det \begin{pmatrix} s & -1 \\ a_2 & s+a_1 \end{pmatrix} \\ (s+15)(s-7) + 120 &= s(s+a_1) + a_2 \\ s^2 + 8s + 15 &= s^2 + a_1 s + a_2 \end{aligned}$$

Example: Converting a Controllable System to CCF

Step 2: Determine desired $\mathcal{C}(\bar{A}, \bar{B})$.

We need to figure out $\bar{A}$ and $\bar{B}$.

For CCF, we must have

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \qquad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have just computed

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore, the new controllability matrix should be

$$\mathcal{C}(\bar{A}, \bar{B}) = [\bar{B} \mid \bar{A}\bar{B}] = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}$$

Example: Converting a Controllable System to CCF

Step 3: Compute T.

Recall: $T = \mathcal{C}(\bar{A}, \bar{B}) \cdot [\mathcal{C}(A, B)]^{-1}$

$$\begin{aligned}
\mathcal{C}(A, B) &= \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix} \\
[\mathcal{C}(A, B)]^{-1} &= \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}^{-1} \\
&= \frac{1}{-1} \begin{pmatrix} -8 & 7 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix} \\
\mathcal{C}(\bar{A}, \bar{B}) &= \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix} \\
T &= \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

In the next lecture, we will see why CCF is so useful.