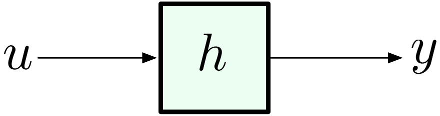
ECE486: Control Systems

- Lecture 3C: Steady-state response, DC gain, and final value theorem

Goal: understand what happens to the response when $t \to \infty$?

DC Gain



Definition: the steady-state value of the step response is called the *DC gain* of the system.

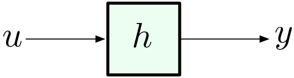
$$\text{DC gain} = y(\infty) = \lim_{t \to \infty} y(t) \qquad \text{for } u(t) = 1(t)$$

In our example above, the step response is

$$y(t) = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

therefore, DC gain $= y(\infty) = 1/2$

Steady-State Value



$$u(t) = 1(t) \qquad U(s) = \frac{1}{s} \qquad \Longrightarrow \qquad Y(s) = \frac{H(s)}{s}$$

— can we compute $y(\infty)$ from $Y(s)$?

Let's look at some examples:

- $Y(s) = \dfrac{1}{s+a}, \; a > 0 \quad$ (pole at $s = -a < 0$)
  $y(t) = e^{-at} \quad \Longrightarrow \quad y(\infty) = 0$
- $Y(s) = \dfrac{1}{s+a}, \; a < 0 \quad$ (pole at $s = -a > 0$)
  $y(t) = e^{-at} \quad \Longrightarrow \quad y(\infty) = \infty$
- $Y(s) = \dfrac{1}{s^2+\omega^2}, \; \omega \in \mathbb{R} \quad$ (poles at $s = \pm j\omega$, purely imaginary)
  $y(t) = \sin(\omega t) \quad \Longrightarrow \quad y(\infty)$ does not exist
- $Y(s) = \dfrac{c}{s} \quad$ (pole at the origin, $s = 0$)
  $y(t) = c1(t) \quad \Longrightarrow \quad y(\infty) = c$

The Final Value Theorem

We can now deduce the Final Value Theorem (FVT):

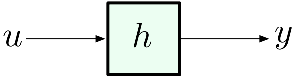
If all poles of $sY(s)$ are *strictly stable* or lie in the *open left half-plane* (OLHP), i.e., have $\text{Re}(s) < 0$, then

$$y(\infty) = \lim_{s \to 0} sY(s).$$

In our examples, multiply $Y(s)$ by s, check poles:

- $Y(s) = \dfrac{1}{s+a} \qquad sY(s) = \dfrac{s}{s+a}$
  if $a > 0$, then $y(\infty) = 0$; if $a < 0$, FVT does not give correct answer
- $Y(s) = \dfrac{1}{s^2+\omega^2} \qquad sY(s) = \dfrac{s}{s^2+\omega^2}$
  poles are purely imaginary (not in OLHP), FVT does not give correct answer
- $Y(s) = \dfrac{c}{s} \qquad sY(s) = c$
  poles at infinity, so $y(\infty) = c$ – FVT gives correct answer

Back to DC Gain



$$\text{Step response:} \qquad Y(s) = \frac{H(s)}{s}$$

— if all poles of $sY(s) = H(s)$ are strictly stable, then

$$y(\infty) = \lim_{s \to 0} H(s)$$

by the FVT.

Example: compute DC gain of the system with transfer function

$$H(s) = \frac{s^2 + 5s + 3}{s^3 + 4s + 2s + 5}$$

All poles of $H(s)$ are strictly stable (we will see this later using the *Routh–Hurwitz criterion*), so

$$y(\infty) = H(s)\Big|_{s=0} = \frac{3}{5}.$$