ECE 486: Control Systems

Lecture 5C: State-Space Models

Key Takeaways

This lecture introduces linear state-space models.

An n^{th}-order linear state-space model expresses the dynamics as a first-order, vector differential equation. It is possible to express as an equivalent n^{th}-order linear ODE.

State-space models have several uses:

- There are different tools for analysis and design of feedback systems based on state-space models.
- They can be used to approximate a nonlinear model by a related linear model.

Linear State-Space Model

An n^{th}-order linear state-space model with one input and one output has the form:

$$
\begin{aligned}
\dot{x}_1(t) &= A_{1,1}x_1(t) + A_{1,2}x_2(t) + \cdots + A_{1,n}x_n(t) + B_1u(t) \\
\dot{x}_2(t) &= A_{2,1}x_1(t) + A_{2,2}x_2(t) + \cdots + A_{2,n}x_n(t) + B_2u(t) \\
&\vdots \qquad\qquad\quad \vdots \qquad\qquad\quad \vdots \\
\dot{x}_n(t) &= A_{n,1}x_1(t) + A_{n,2}x_2(t) + \cdots + A_{n,n}x_n(t) + B_nu(t) \\
y(t) &= C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t) + Du(t) \\
&\text{IC: } x_1(0) = x_{1,0}; \ldots ; x_n(0) = x_{n,0}
\end{aligned}
$$

This is n coupled first-order ODEs. We can express this compactly using matrices and vectors:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t) \\
&\text{IC: } x(0) = x_0
\end{aligned}
$$

where $x \in \mathbb{R}^n$ is the state

ODE to State-Space

Consider the third-order ODE:

$$y^{[3]}(t) + 0.2\ddot{y}(t) - 0.3\dot{y}(t) + 7y(t) = -0.4\ddot{u}(t) + 5\dot{u}(t) + 11u(t)$$

Recall that we can re-write this to avoid differentiating u:

$$w^{[3]}(t) + 0.2\ddot{w}(t) - 0.3\dot{w}(t) + 7w(t) = u(t)$$
$$y(t) = -0.4\ddot{w}(t) + 5\dot{w}(t) + 11w(t)$$

Define the state-variables:

$$x_1 := w,\ x_2 := \dot{w},\ x_3 := \ddot{w}$$

ODE to State-Space

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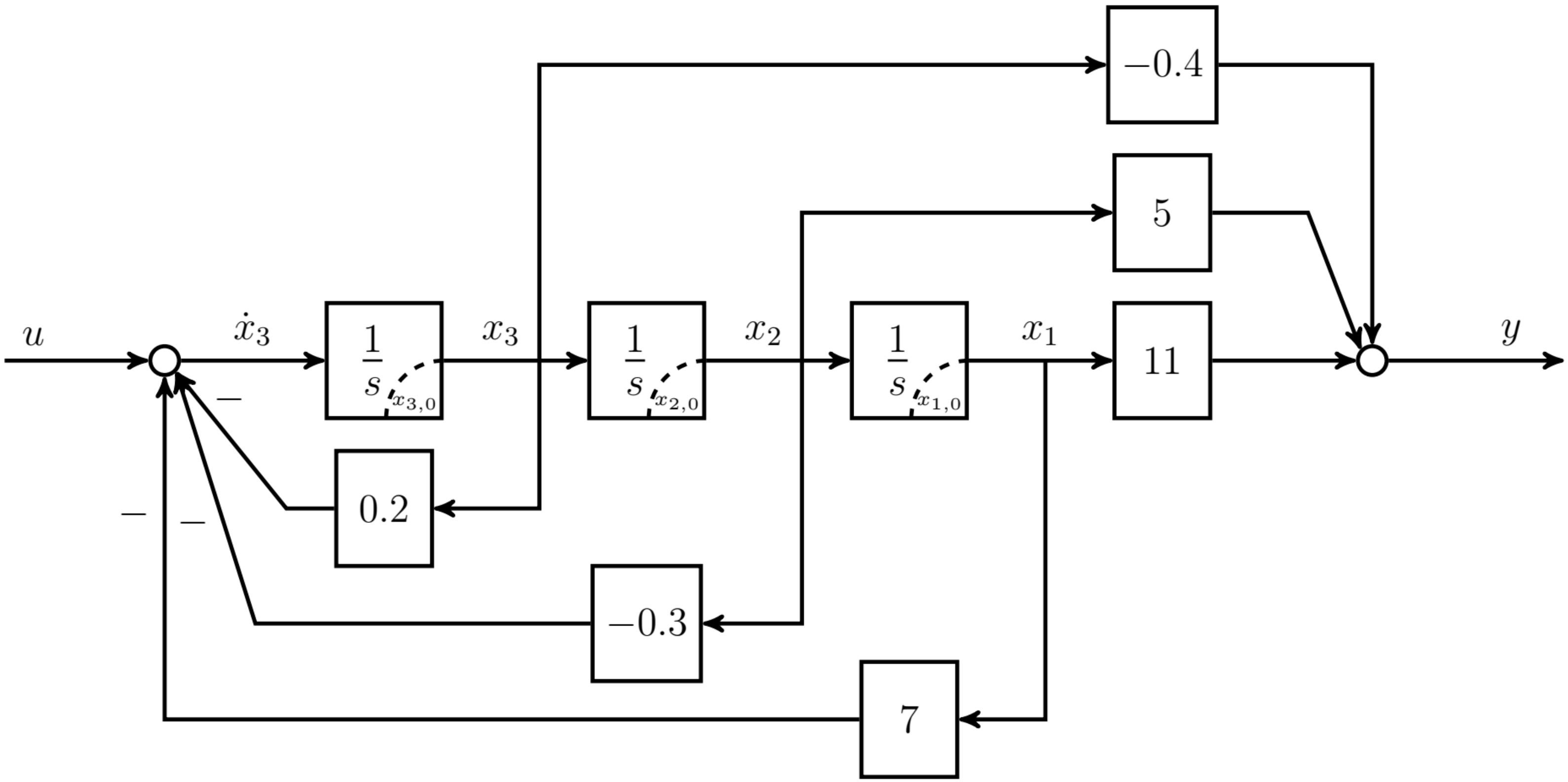
A block diagram is shown below.

The states are the outputs of the integrators.



ODE to State-Space

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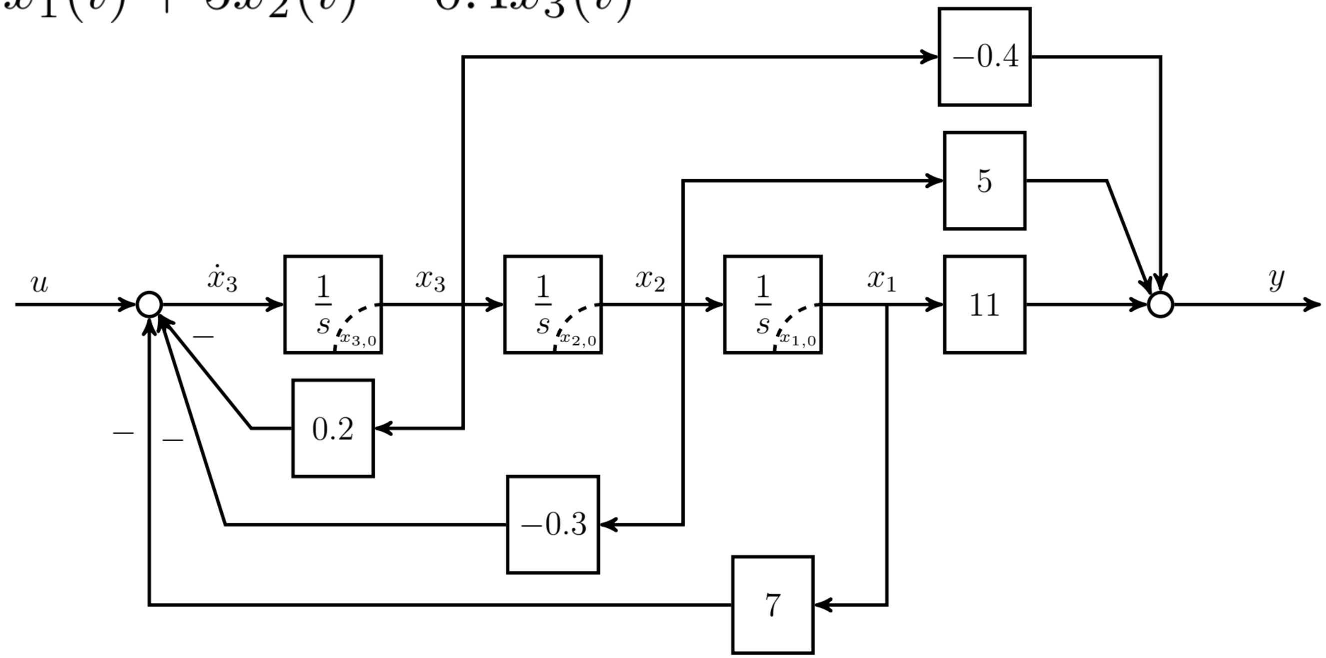
The derivatives satisfy:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3 \text{ and } \dot{x}_3(t) = -7x_1(t) + 0.3x_2(t) - 0.2x_3(t) + u(t)$$

The output satisfies:

$$y(t) = 11x_1(t) + 5x_2(t) - 0.4x_3(t)$$



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This gives the state-space model

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 0.3 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 11 & 5 & -0.4 \end{bmatrix}, \quad D = 0$$

The state-space model is not unique. (We can define a new set state *z=Tx* where *T* is a non-singular matrix.)

State-Space to ODE

Consider an nth-order state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Roughly, replace the differentiation with "s":

$$sX(s) = AX(s) + BU(s) \quad \Rightarrow X(s) = (sI - A)^{-1}BU(s)$$

Substitute into the output equation:

$$Y(s) = CX(s) + DU(s) \quad \Rightarrow Y(s) = \left[C(sI - A)^{-1}B + D\right]U(s)$$
$$\Rightarrow G(s) = C(sI - A)^{-1}B + D$$

This is useful conceptually, but it does not provide the ODE coefficients associated with numerator/denominator polynomials.

These can be obtained with some linear algebra results but we will rely on numerical tools, e.g. Matlab.

Example

Consider the third-order ODE:

$$y^{[3]}(t) + 0.2\ddot{y}(t) - 0.3\dot{y}(t) + 7y(t) = -0.4\ddot{u}(t) + 5\dot{u}(t) + 11u(t)$$

```
>> A=[0 1 0; 0 0 1; -7 0.3 -0.2];
>> B=[0;0;1]; C=[11 5 -0.4]; D=0;
>> G=ss(A,B,C,D);

% Comment: tf() converts G from SS to TF form. Note that we
% recover the TF for the original 3rd-order ODE.
>> tf(G)
ans =
     -0.4 s^2 + 5 s + 11
  -------------------------
  s^3 + 0.2 s^2 - 0.3 s + 7
```

Example

Consider the third-order ODE:

$$y^{[3]}(t) + 0.2\ddot{y}(t) - 0.3\dot{y}(t) + 7y(t) = -0.4\ddot{u}(t) + 5\dot{u}(t) + 11u(t)$$

```
% We can also construct the original TF and convert from TF to SS.
>> G2 = tf([-0.4 5 11],[1 0.2 -0.3 7]);     % Construct original TF
>> G3 = ss(G2);                             % ss() converts G2 from TF to SS form

% Note that A3 is not the same as A given above. This is due to
% the non-uniqueness of state-space models, i.e. both G and G3
% represent the same dynamics but with different state matrices.
>> [A3,B3,C3,D3]=ssdata(G3);
>> A3
A3 =
   -0.2000    0.1500   -1.7500
    2.0000         0         0
         0    2.0000         0
```