Plan of the Lecture

- Review: transient and steady-state response; DC gain and the FVT
- Today's topic: prototype 2nd-order system; Transient response specifications

Goal: start analyzing a prototype 2nd-order system; develop formulas and intuition for various features of the transient response: rise time, overshoot, settling time..

Prototype 2nd-Order System

So far, we have only seen transfer functions that have either real poles or purely imaginary poles:

$$\frac{1}{s+a}, \qquad \frac{1}{(s+a)(s+b)}, \qquad \frac{1}{s^2+\omega^2}$$

We also need to consider the case of *complex poles*, i.e., ones that have $\text{Re}(s) \neq 0$ and $\text{Im}(s) \neq 0$.

For now, we will only look at *second-order systems*, but this will be sufficient to develop some nontrivial intuition (dominant poles).

Plus, you will need this for Lab 1.

Prototype 2nd-Order System

Consider the following transfer function:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Comments:

- $\zeta > 0, \omega_n > 0$ are arbitrary parameters
- the denominator is a general 2nd-degree monic polynomial, just written in a weird way
- $H(s)$ is normalized to have DC gain $= 1$ (provided DC gain exists)

Prototype 2nd-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

By the quadratic formula, the poles are:

$$\begin{aligned} s &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right) \end{aligned}$$

The nature of the poles changes depending on ζ:

- $\zeta > 1$ both poles are real and negative
- $\zeta = 1$ one negative pole
- $\zeta < 1$ two complex poles with negative real parts

$$s = -\sigma \pm j\omega_d$$

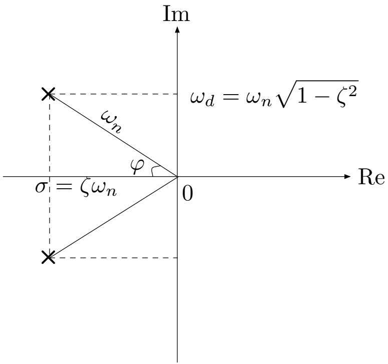
$$\text{where} \qquad \sigma = \zeta\omega_n,\ \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Prototype 2nd-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad \zeta < 1$$

The poles are

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$



Note that

$$\begin{aligned}\sigma^2 + \omega_d^2 &= \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2 \\ &= \omega_n^2 \\ \cos\varphi &= \frac{\zeta\omega_n}{\omega_n} = \zeta\end{aligned}$$

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\}$$
$$= \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t) \qquad \text{(table, \# 20)}$$

- Step response:

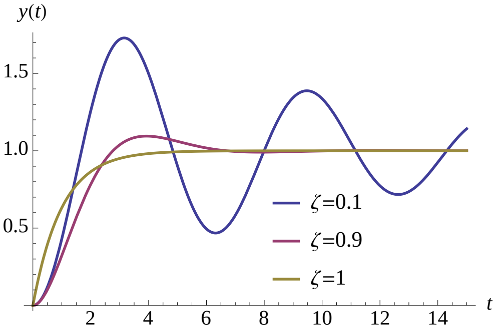
$$\mathscr{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{\sigma^2 + \omega_d^2}{s[(s+\sigma)^2 + \omega_d^2]}\right\}$$
$$= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) \qquad \text{(table, \#21)}$$

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)



The parameter ζ is called the *damping ratio*

- $\zeta > 1$: system is overdamped
- $\zeta < 1$: system is underdamped
- $\zeta = 0$: no damping ($\omega_d = \omega_n$)

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)

We will see that the parameters ζ and ω_n determine certain important features of the transient part of the above step response.

We will also learn how to pick ζ and ω_n in order to *shape* these features according to given *specifications*.

Transient Response Specifications: Rise Time

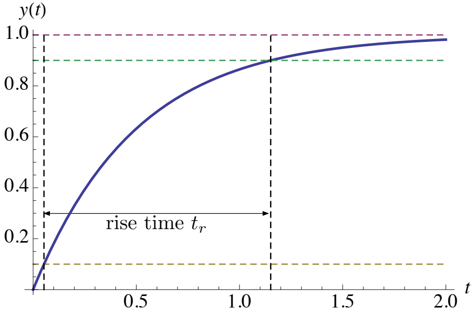
Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0 \qquad \text{(stable pole)}$$

DC gain $= 1$ (by FVT)

Step response: $$Y(s) = \frac{H(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

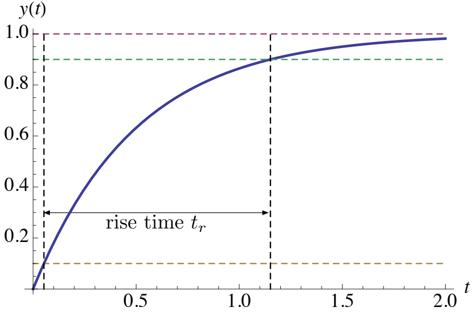
$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 1(t) - e^{-at}$$



Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

Rise Time

Step response: $y(t) = 1(t) - e^{-at}$



Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

In this example, it is easy to compute t_r analytically:

$$1 - e^{-at_{0.1}} = 0.1 \qquad e^{-at_{0.1}} = 0.9 \qquad t_{0.1} = -\frac{\ln 0.9}{a}$$

$$1 - e^{-at_{0.9}} = 0.9 \qquad e^{-at_{0.9}} = 0.1 \qquad t_{0.9} = -\frac{\ln 0.1}{a}$$

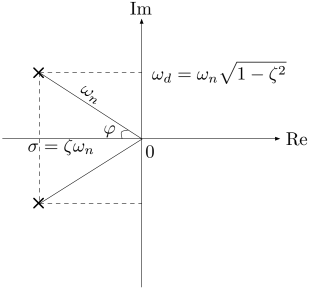
$$t_r = t_{0.9} - t_{0.1} = \frac{\ln 0.9 - \ln 0.1}{a} = \frac{\ln 9}{a} \approx \frac{2.2}{a}$$

Transient Response Specs

Now let's consider the more interesting case: *2nd-order response*

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

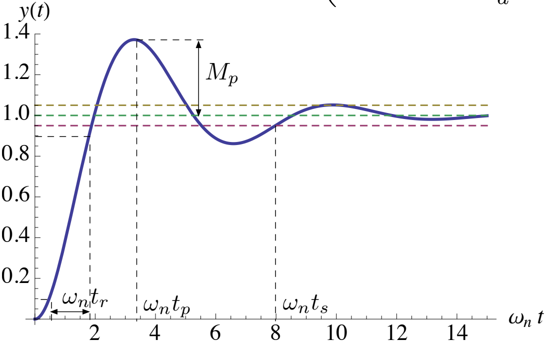
where $\sigma = \zeta\omega_n \; \omega_d = \omega_n\sqrt{1-\zeta^2} \qquad (\zeta < 1)$



Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

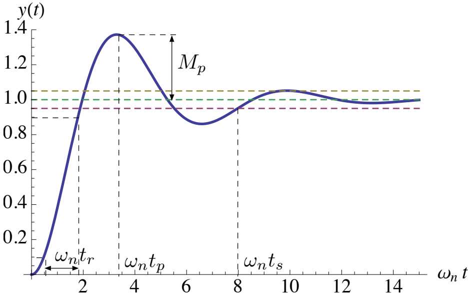
Transient-Response Specs

Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$



- rise time t_r — time to get from $0.1y(\infty)$ to $0.9y(\infty)$
- overshoot M_p and peak time t_p
- settling time t_s — first time for transients to decay to within a specified small percentage of $y(\infty)$ and stay in that range (we will usually worry about 5% settling time)

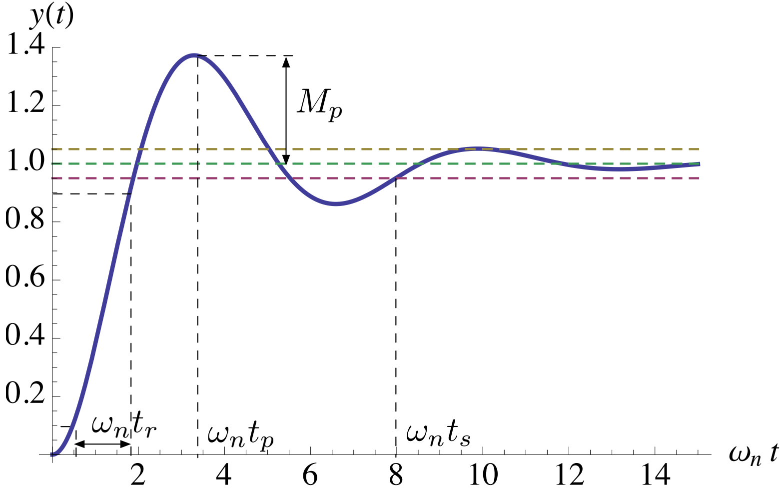
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

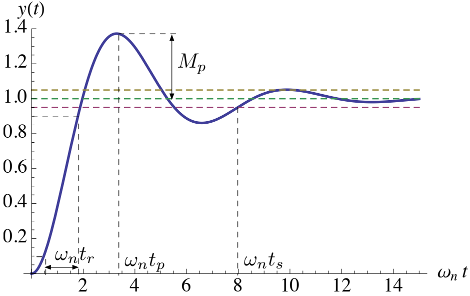
- t_r small
- M_p small
- t_p small
- t_s small

Trade-offs among specs: decrease $t_r \longrightarrow$ increase M_p, etc.



$y(t)$
1.4
1.2
1.0
0.8
0.6
0.4
0.2
M_p
$\omega_n t_r$
$\omega_n t_p$
$\omega_n t_s$
2
4
6
8
10
12
14
$\omega_n\, t$

Formulas for TD Specs: Rise Time

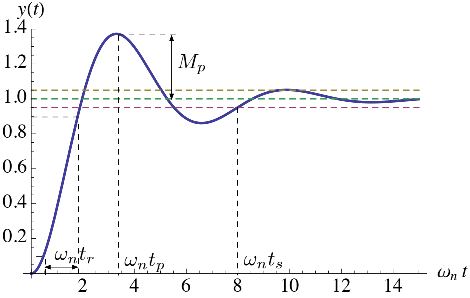


Rise time t_r — hard to calculate analytically.
Empirically, on the normalized time scale ($t \to \omega_n t$), rise times are *approximately* the same

$$w_n t_r \approx 1.8 \qquad \text{(exact for } \zeta = 0.5\text{)}$$

So, we will work with $t_r \approx \dfrac{1.8}{\omega_n}$ (good approx. when $\zeta \approx 0.5$)

Formulas for TD Specs: Overshoot & Peak Time



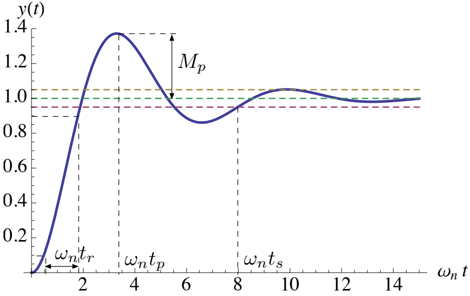
t_p is the *first time* $t > 0$ when $y'(t) = 0$

$$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

$$y'(t) = \left(\frac{\sigma^2}{\omega_d} + \omega_d\right) e^{-\sigma t}\sin(\omega_d t) = 0 \text{ when } \omega_d t = 0, \pi, 2\pi, \ldots$$

so $t_p = \dfrac{\pi}{\omega_d}$

Formulas for TD Specs: Overshoot & Peak Time



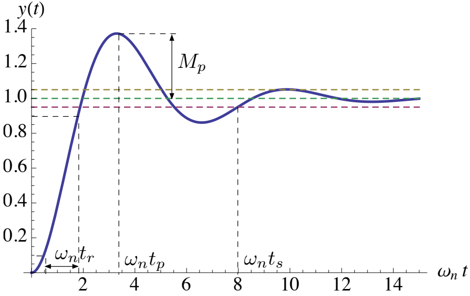
We have just computed $t_p = \dfrac{\pi}{\omega_d}$

To find M_p, plug this value into $y(t)$:

$$M_p = y(t_p) - 1 = -e^{-\frac{\sigma\pi}{\omega_d}} \left(\cos\left(\omega_d \frac{\pi}{\omega_d}\right) + \frac{\sigma}{\omega_d} \sin\left(\omega_d \frac{\pi}{\omega_d}\right)\right)$$

$$= \exp\left(-\frac{\sigma\pi}{\omega_d}\right) = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \qquad \text{— exact formula}$$

Formulas for TD Specs: Settling Time



$t_s = \min\left\{ t > 0 : \dfrac{|y(t') - y(\infty)|}{y(\infty)} \le 0.05 \text{ for all } t' \ge t \right\}$ (here, $y(\infty) = 1$)

$$|y(t) - 1| = e^{-\sigma t} \left| \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right|$$

here, $e^{-\sigma t}$ is what matters (sin and cos are bounded between ± 1), so $e^{-\sigma t_s} \le 0.05$ this gives $t_s = -\dfrac{\ln 0.05}{\sigma} \approx \dfrac{3}{\sigma}$

Formulas for TD Specs

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

$$t_r \approx \frac{1.8}{\omega_n}$$

$$t_p = \frac{\pi}{\omega_d}$$

$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$t_s \approx \frac{3}{\sigma}$$

TD Specs in Frequency Domain

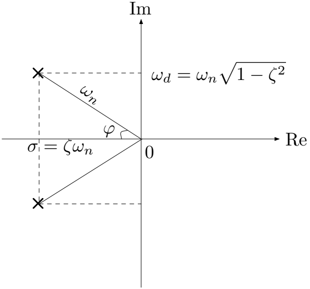
We want to *visualize* time-domain specs in terms of *admissible pole locations* for the 2nd-order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

where $\sigma = \zeta\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Step response: $y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$



$$\omega_n^2 = \sigma^2 + \omega_d^2$$

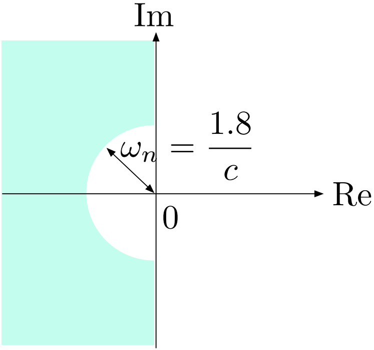
$$\zeta = \cos\varphi$$

Rise Time in Frequency Domain

Suppose we want $t_r \leq c$ (c is some desired given value)

$$t_r \approx \frac{1.8}{\omega_n} \leq c \qquad \Longrightarrow \qquad \omega_n \geq \frac{1.8}{c}$$

Geometrically, we want poles to lie in the shaded region:



(recall that ω_n is the *magnitude of the poles*)

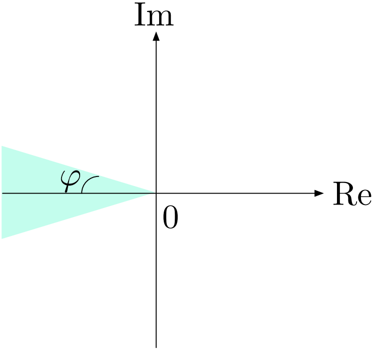
Overshoot in Frequency Domain

Suppose we want $M_p \leq c$

$$M_p = \underbrace{\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}_{\text{decreasing function}} \leq c$$

— need large damping ratio

Geometrically, we want poles to lie in the shaded region:



$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\omega_n \zeta}{\omega_n \sqrt{1-\zeta^2}} = \frac{\sigma}{\omega_d} = \cot\varphi$$

— need φ to be small

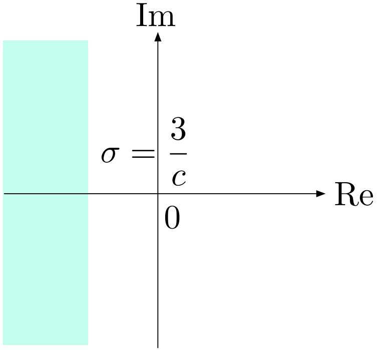
Intuition: good damping → good decay in 1/2 period

Settling Time in Frequency Domain

Suppose we want $t_s \leq c$

$$t_s \approx \frac{3}{\sigma} \leq c \qquad \Longrightarrow \qquad \sigma \geq \frac{3}{c}$$

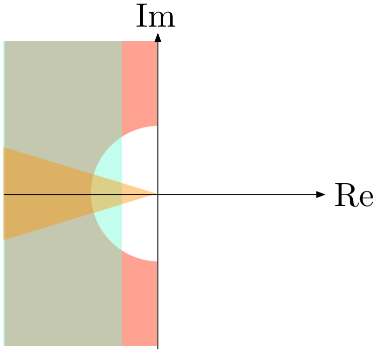
Want poles to be sufficiently fast (large enough magnitude of real part):



Intuition: poles far to the left → transients decay faster → smaller t_s

Combination of Specs

If we have specs for any combination of t_r, M_p, t_s, we can easily relate them to allowed pole locations:



The shape and size of the region for admissible pole locations will change depending on which specs are more severely constrained.

This is very appealing to engineers: easy to visualize things, no such crisp visualization in time domain.

But: not very rigorous, and also only valid for our prototype 2nd-order system, which has only 2 poles and no zeros ...