Plan of the Lecture

- Review: design using Root Locus; dynamic compensation; PD and lead control
- Today's topic: PI and lag control; introduction to frequency-response design method

Goal: wrap up lead and lag control; start looking at frequency response as an alternative methodology for control systems design.

Reading: FPE, Sections 5.1–5.4, 6.1

Recap: Lead & Lag Compensators

Consider a general controller of the form

$$K\frac{s+z}{s+p} \qquad \text{— } K, z, p > 0 \text{ are design parameters}$$

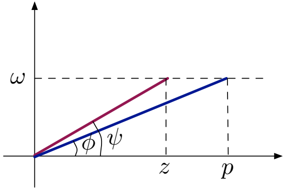
Depending on the relative values of z and p, we call it:

- a lead compensator when $z < p$
- a lag compensator when $z > p$

Why the name "lead/lag?" — think frequency response

$$\angle\frac{j\omega + z}{j\omega + p} = \angle(j\omega + z) - \angle(j\omega + p) = \psi - \phi$$

- if $z < p$, then $\psi - \phi > 0$ (phase lead)
- if $z > p$, then $\psi - \phi < 0$ (phase lag)



Approximate PI via Dynamic Compensation

PI control achieves the objective of stabilization and perfect steady-state tracking of constant references; however, just as with PD earlier, we want a *stable controller*.

Here's an idea:

$$\text{replace } K\frac{s+1}{s} \quad \text{by} \quad K\frac{s+1}{s+p}, \text{ where } p \text{ is small}$$

More generally, if $z = K_{\mathrm{I}}/K_{\mathrm{P}}$, then

$$\text{replace } K\frac{s+z}{s} \quad \text{by} \quad K\frac{s+z}{s+p}, \text{ where } p < z$$

This is lag compensation (or lag control)!

We use lag controllers as dynamic compensators for approximate PI control.

Approximate PI via Lag Compensation

$$G_c(s) = K\frac{s+z}{s+p},\ p < z \qquad\qquad G_p(s) = \frac{1}{s-1}$$

How good is this controller?

Tracking a constant reference: assuming closed-loop stability, the FVT gives

$$e(\infty) = \frac{1}{1+G_c(s)G_p(s)}\Big|_{s=0} = \frac{1}{1+K\frac{s+z}{(s+p)(s-1)}}\Big|_{s=0} = \frac{1}{1-\frac{Kz}{p}}$$

Check for stability: no RHP poles for $\dfrac{1}{1+G_c(s)G_p(s)}$

$$(s+p)(s-1) + K(s+z) = 0$$

$$s^2 + (K+p-1)s + Kz - p = 0$$

Conditions for stability: $K > 1-p,\ Kz > p$

Approximate PI via Lag Compensation

Tracking a constant reference: if the stability conditions

$$K > 1 - p, \quad Kz > p$$

are satisfied, then the steady-state error is

$$e(\infty) = \frac{1}{1 - \frac{Kz}{p}}$$

— this will be close to zero (and negative) if $\frac{Kz}{p}$ is large.

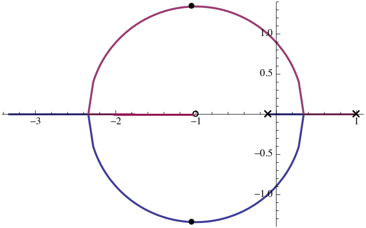
Lag compensation *does not* give perfect tracking (indeed, it does not change system type), but we can get as good a tracking as we want by playing with K, z, p. On the other hand, unlike PI, lag compensation gives a stable controller.

Effect of Lag Compensation on Root Locus

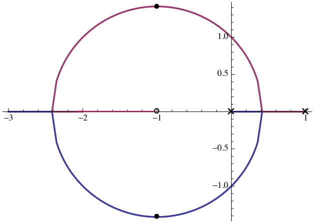
$$L(s) = \frac{s+1}{(s+p)(s-1)}$$

Intuition: By choosing p very close to zero, we can make the root locus arbitrarily close to PI root locus (stable for large enough K). Let's check:

Try $p = 0.1$



Compare to PI root locus:



What do we see? Compared to PD vs. lead, there is no qualitative change in the shape of RL, since we are not changing #(poles) or #(zeros).

More Pole Placement

As before, we can choose z_{lag} for a fixed p_{lag} (or vice versa) based on desired pole locations.

The procedure is exactly the same as the one we used with lead. (In fact, depending on the pole locations, we may end up with either lead or lag.)

Main technique: select parameters to satisfy the phase condition (points on RL must be such that $\angle L(s) = 180^{\circ}$).

Caveat: may not always be possible!

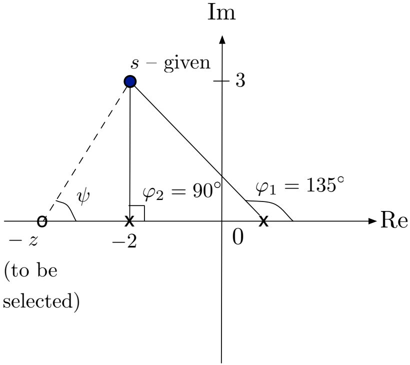
Pole Placement via RL

Let $G_p(s) = \dfrac{1}{s-1}, \qquad G_c(s) = K\dfrac{s+z}{s+p}$

Problem: given $p = 2$, find K and z to place poles at $-2 \pm 3j$.

Desired characteristic polynomial:

$$(s+2)^2 + 9 = s^2 + 4s + 13, \qquad \text{damping ratio } \zeta = \frac{2}{\sqrt{13}} \approx 0.555$$

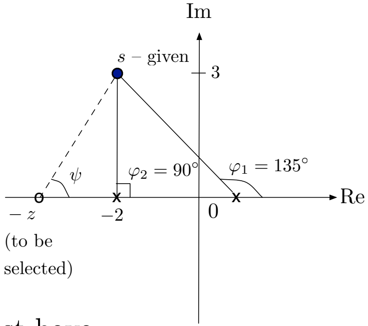


Must have

$$\underbrace{\psi}_{\text{angle from } s \text{ to zero}} - \sum_i \underbrace{\varphi_i}_{\text{angles from } s \text{ to poles}} = 180^\circ$$

So, we want $\psi = 180^\circ + \sum_i \varphi_i$

Pole Placement via RL



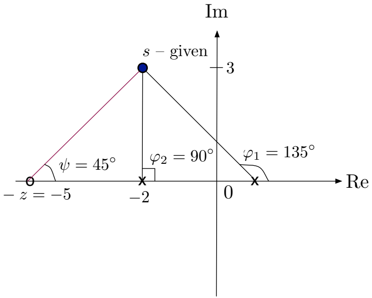
We have
$\varphi_1 = 135°,$
$\varphi_2 = 90°$

We want $\psi = 180° + \sum_i \varphi_i$

Must have

$$\psi = 180° + 135° + 90°$$
$$= 405°$$
$$= 45° \mod 360°$$

Thus, we should have $z = -5$



Let $G_p(s) = \frac{1}{s-1}, \qquad G_c(s) = K\frac{s+z}{s+p}$

Problem: given $p = 2$, find z to place poles at $-2 \pm 3j$.

Solution:

- we already found that we need $z = 5$
- resulting characteristic polynomial:

$$(s-1)(s+2) + K(s+5)$$
$$s^2 + (K+1)s + 5K - 2$$

- compare against desired characteristic polynomial:

$$s^2 + 4s + 13 \qquad \Longrightarrow \qquad K + 1 = 4, \ 5K - 2 = 13$$

  so we need $K = 3$

- compute s.s. tracking error: $\left| \frac{1}{1 - \frac{Kz}{p}} \right| = \frac{1}{6.5} \approx 15\%$

Story So Far

PD control:

- provides stability, allows to shape transient response specs
- replace noncausal D-controller Ks with a causal, stable lead controller $K\frac{s+z}{s+p}$, where $p > z$
- this introduces a zero in LHP (at $-z$), pulls the root locus into LHP
- shape of RL differs depending on how large p is

PI control:

- provides stability and perfect steady-state tracking of constant references
- replace unstable I-controller K/s with a stable lag controller $K\frac{s+z}{s+p}$, where $p < z$
- this does not change the shape of RL compared to PI

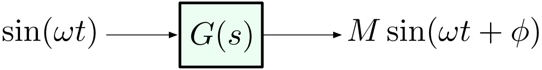
What About PID Control?

Obvious solution — combine lead *and* lag compensation.

We will develop this further in homework and later in the course using frequency-response design methods — which are the subject of several lectures, starting with today's.

The Frequency-Response Design Method

Recall the frequency-response formula:



where $M = M(\omega) = |G(j\omega)|$ and $\phi = \phi(\omega) = \angle G(j\omega)$

Derivation:

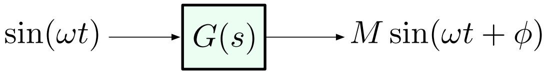
1. $u(t) = e^{st} \longmapsto y(t) = G(s)e^{st}$
2. Euler's formula: $\sin(\omega t) = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}$
3. By linearity,

$$\sin(\omega t) \longmapsto \frac{G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}}{2j} \quad G(j\omega) = M(\omega)e^{j\phi(\omega)}$$

$$= \frac{M(\omega)e^{j(\omega t + \phi(\omega))} - M(\omega)e^{-j(\omega t + \phi(\omega))}}{2j}$$

$$= M(\omega)\sin(\omega t + \phi(\omega))$$

The Frequency-Response Design Method



where $M = M(\omega) = |G(j\omega)|$ and $\phi = \phi(\omega) = \angle G(j\omega)$

Let's apply this formula to our prototype 2nd-order system:

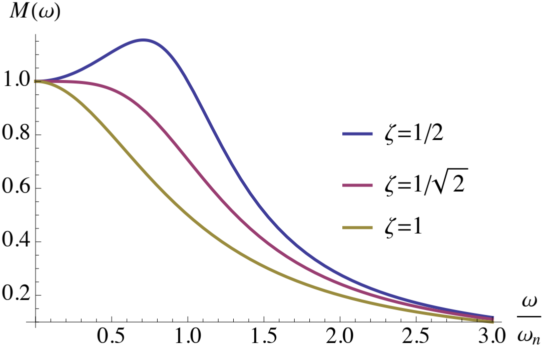
$$\begin{aligned}
G(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
M(\omega) = |G(j\omega)| &= \left| \frac{\omega_n^2}{-\omega^2 + 2j\zeta\omega_n\omega + \omega_n^2} \right| \\
&= \left| \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{\omega}{\omega_n}j} \right| \\
&= \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}
\end{aligned}$$

The Frequency-Response Design Method

For our prototype 2nd-order system:

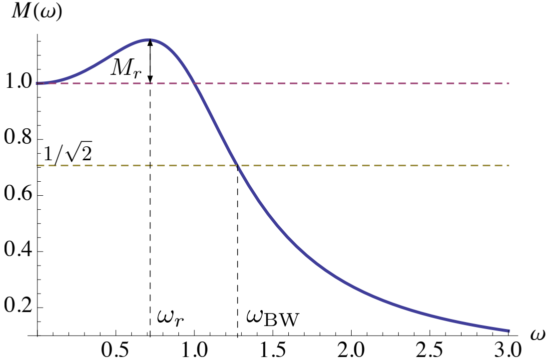
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$M(\omega) = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{1 + (4\zeta^2 - 2)\left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{\omega}{\omega_n}\right)^4}}$$



Frequency Response Parameters

Here is a typical frequency response magnitude plot:

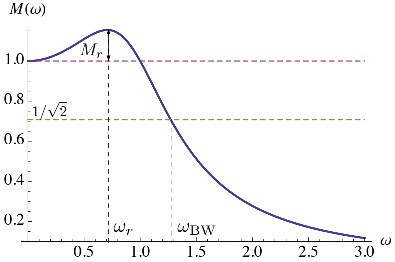


ω_r – resonant frequency

M_r – resonant peak

ω_{BW} – bandwidth

Frequency Response Parameters



We can get the following formulas using calculus:

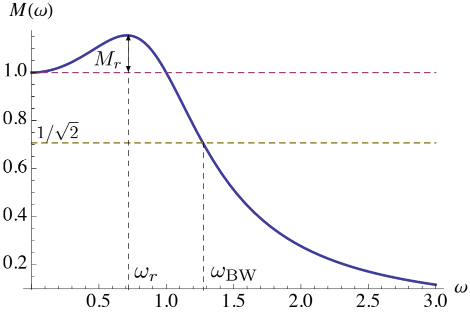
$$\begin{cases} \omega_r = \omega_n \sqrt{1-2\zeta^2} \\ M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} - 1 \end{cases} \qquad \left(\text{valid for } \zeta < \frac{1}{\sqrt{2}}; \text{ for } \zeta \geq \frac{1}{\sqrt{2}}, \ \omega_r = 0\right)$$

$$\omega_{\text{BW}} = \omega_n \underbrace{\sqrt{(1-2\zeta^2) + \sqrt{(1-2\zeta^2)^2 + 1}}}_{=1 \text{ for } \zeta = 1/\sqrt{2}}$$

— so, if we know $\omega_r, M_r, \omega_{\text{BW}}$, we can determine ω_n, ζ and hence the time-domain specs (t_r, M_p, t_s)

Frequency Response & Time-Domain Specs

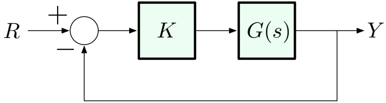
All information about time response is also encoded in frequency response!!



$$\text{small } M_r \longleftrightarrow \text{better damping}$$

$$\text{large } \omega_{\mathrm{BW}} \longleftrightarrow \text{large } \omega_n \longleftrightarrow \text{smaller } t_r$$

Frequency-Response Design Method: Main Idea



Two-step procedure:

1. Plot the frequency response of the *open-loop* transfer function $KG(s)$ [or, more generally, $D(s)G(s)$], at $s = j\omega$
2. See how to relate this open-loop frequency response to closed-loop behavior.

We will work with two types of plots for $KG(j\omega)$:

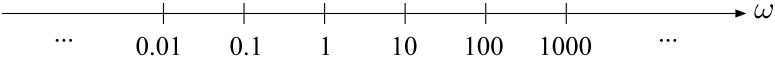
1. Bode plots: magnitude $|KG(j\omega)|$ and phase $\angle KG(j\omega)$ vs. frequency ω (could have seen it earlier, in ECE 342)
2. Nyquist plots: $\text{Im}\big(KG(j\omega)\big)$ vs. $\text{Re}\big(K(j\omega)\big)$ [Cartesian plot in s-plane] as ω ranges from $-\infty$ to $+\infty$

Note on the Scale

Horizontal (ω) axis:

we will use *logarithmic scale* (base 10) in order to display a wide range of frequencies.

Note: we will still mark the values of ω, *not* $\log_{10} \omega$, on the axis, but the *scale* will be logarithmic:



Equal intervals on log scale correspond to decades in frequency.

Note on the Scale

Vertical axis on magnitude plots:

we will also use logarithmic scale, just like the frequency axis.

Reason:

$$|(M_1 e^{j\phi_1})(M_2 e^{j\phi_2})| = M_1 \cdot M_2$$
$$\log(M_1 M_2) = \log M_1 + \log M_2$$

— this means that we can simply *add* the graphs of $\log M_1(\omega)$ and $\log M_2(\omega)$ to obtain the graph of $\log\big(M_1(\omega)M_2(\omega)\big)$, and graphical addition is easy.

Decibel scale:

$$(M)_{\mathrm{dB}} = 20\log_{10} M \qquad \text{(one decade} = 20\,\mathrm{dB)}$$

Note on the Scale

Vertical axis on phase plots:

we will plot the phase on the usual (linear) scale.

Reason:

$$\begin{aligned}\angle\Big((M_1 e^{j\phi_1})(M_2 e^{j\phi_2})\Big) &= \angle\Big(M_1 M_2 e^{j(\phi_1+\phi_2)}\Big)\\ &= \phi_1 + \phi_2\end{aligned}$$

— this means that we can simply *add* the phase plots for two transfer functions to obtain the phase plot for their product.

Scale Convention for Bode Plots

	magnitude	phase
horizontal scale	log	log
vertical scale	log	linear

Advantage of the scale convention: we will learn to do Bode plots by starting from simple factors and then building up to general transfer functions by considering products of these simple factors.