SAMPLE FINAL EXAM

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1. Let the plant be given by the equations

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -2x_1 + u\\ y &= x_1 + x_2\end{aligned}$$

a) Draw a block diagram of this system. The blocks you can use are integrators, gains, and summing junctions. Make sure to label all signals.

b) Derive the plant transfer function.

c) Let the controller be given by the transfer function

$$\frac{5}{s+3}$$

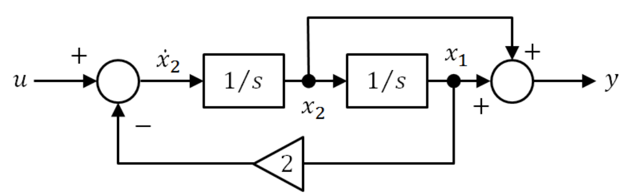
Write down two different state-space realizations of this controller, both of them stable.

d) Consider now the closed-loop system obtained by interconnecting the plant from part a) and the controller given by one of your realizations (choose any one) from part c). By "interconnecting" we mean that the plant's output is the controller's input, the controller's output is minus the plant's input (negative feedback configuration), and the state of the overall closed-loop system consists of the plant's state and the controller's state. Write down the differential equations describing this closed-loop system.

e) Is it true that the state of the closed-loop system decays to 0 from arbitrary initial condition as time goes to ∞?

Solution:

(a) Block Diagram



(b)

$$\begin{aligned}sx_1 &= x_2\\ s^2x_1 &= sx_2 = -2x_1 + u\end{aligned}$$

$$\Rightarrow X_1(s) = \frac{1}{s^2+2}U(s)$$

$$Y(s) = X_1(s) + X_2(s) = \frac{s+1}{s^2+2}U(s)$$

so

$$G(s) = \frac{s+1}{s^2+2}$$

<u>Or</u>

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

$$\begin{aligned} (Is - A)^{-1} &= \begin{pmatrix} s & -1 \\ 2 & s \end{pmatrix}^{-1} \\ &= \frac{1}{s^2+2} \begin{pmatrix} s & 1 \\ -2 & s \end{pmatrix} \end{aligned}$$

$$\begin{aligned} G(s) &= C\,(Is - A)^{-1}\,B \\ &= \frac{1}{s^2+2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} s & 1 \\ -2 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{s+1}{s^2+2} \end{aligned}$$

(c) realization 1

$$\begin{aligned} \dot{x} &= -3x + u \\ y &= 5x \end{aligned}$$

realization 2

$$\begin{aligned} \dot{x} &= -3x + 5u \\ y &= x \end{aligned}$$

realization 3

$$\begin{aligned} \dot{x}_1 &= -3x_1 + 5u \\ \dot{x}_2 &= -x_2 \\ y &= x_1 \end{aligned}$$

Note: There are an infinite number of realizations.

(d) Choosing realization 1 for controller, relabeling signals as

$$\begin{aligned} \dot{x}_3 &= -3x_3 + y \\ u &= 5x_3 \end{aligned}$$

gives closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 + 5x_3 \\ \dot{x}_3 &= x_1 + x_2 - 3x_3 \end{aligned}$$

(e) Yes! Closed-loop matrix is

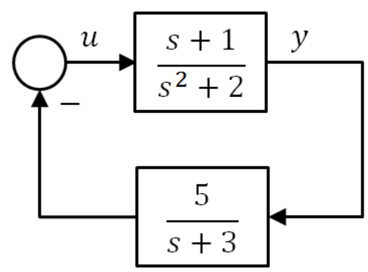
$$A_{cloop} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & -5 \\ 1 & 1 & -3 \end{pmatrix}$$

charact. eq'n is

$$\det(Is - A) = \det\begin{pmatrix} s & -1 & 0 \\ 2 & s & 5 \\ -1 & -1 & s+3 \end{pmatrix}$$
$$= s(s^2+3s+5) + 2s + 6 + 5$$
$$= s^3 + 3s^2 + 7s + 11$$

By Routh-Hurwitz, roots are LHP
(If you forgot to add the minus sign to the control input, then the answer is "No".)

<u>Or</u> can do it using transfer functions:



Charact. eq'n is

$$1 + \frac{s+1}{s^2+2} \cdot \frac{5}{5+3} = 0$$
$$\Rightarrow (s^2+2)(s+3) + (s+1)5 = 0$$
$$\Rightarrow s^3 + 3s^2 + 7s + 11 = 0$$

✓stable

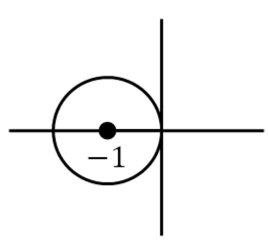
2. Suppose that a plant with transfer function $G(s)$ is put in standard negative feedback configuration with gain $K = 1$, and suppose we know that this results in a stable closed-loop system (in other words, we know that the gain $K = 1$ is stabilizing). Further, suppose we know that the Nyquist plot of $G(s)$ *never enters inside* the disk in the complex plane with radius 1 centered at the point -1.

a) Show that the gain margin is $+\infty$, in the sense that any gain $K > 1$ is also stabilizing.

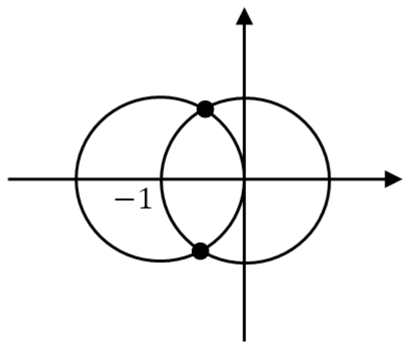
b) Show that the phase margin is at least $60°$.

Solution:

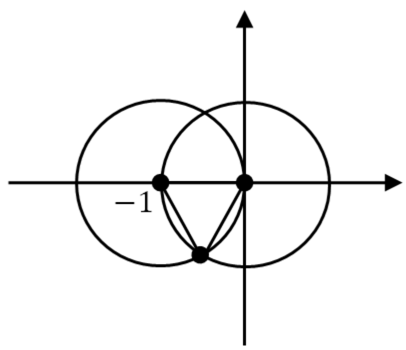
(a) Nyquist plot doesn't cross the real axis anywhere on $(-1, 0)$
$\Rightarrow$ for $k > 1$, the Nyquist plot of $KG(s)$ will never cross the point $-1 \Rightarrow GM = \infty$



(b) Need to look at the phase of points on Nyquist plot that lie on unit circle centered at 0. The smallest possible difference from 180° is at marked points. They satisfy $(s+1)^2+y^2=1$, $s^2+y^2=1$. Solving, we get $x=-1/2$, $y=\pm\sqrt{3}/2 \Rightarrow PM=60°$.



Or geometrically, equilateral triangle $\Rightarrow PM=60°$.



3. Consider the system

$$\dot{x}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}x+\begin{pmatrix}1\\ 0\end{pmatrix}u$$

a) Is it possible to place closed-loop eigenvalues at arbitrary locations by full-state feedback? If yes, explain how. If not, explain what closed-loop eigenvalues are achievable and why others are not.

b) *For extra credit:* Let the initial condition be $x(0)=\binom{0}{1}$, and for each $t\geq 0$ let R_t be the set of all states $x(t)$ that can be reached from this initial condition after time t by applying arbitrary controls. Describe what these sets R_t look like. Your answer may involve a picture, but it must be supported by mathematical explanations. (Hint: you can use the fact that for a *controllable* linear system, any initial state can be steered to any other state in arbitrary time by applying some control.)

Solution:

(a) System is not controllable $\Rightarrow$ cannot get arbitrary poles. With $u=\begin{pmatrix}k_1 & k_2\end{pmatrix}x$ we have

$$\begin{aligned}A-BK &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}1\\ 0\end{pmatrix}\begin{pmatrix}k_1 & k_2\end{pmatrix}\\ &= \begin{pmatrix}1-k_1 & k_2\\ 0 & 1\end{pmatrix},\end{aligned}$$

which has eigenvalues of $1-k_1$ (can be anything) and 1 (cannot be changed)

(b) $\dot{x}_1 = x_1 + u$ controllable $\Rightarrow$ all $x_1(t)$ are possible
$\dot{x}_2 = x_2$ no control $\Rightarrow x_2(t) = e^t x_2(0) = e^t$
So, R_t is horizontal line through $\begin{pmatrix} 0 \\ e^t \end{pmatrix}$, i.e.,

$$R_t = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \in R, x_2 = e^t \right\}$$