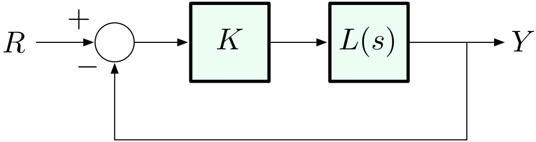
ECE486: Control Systems

- Lecture 11B: Root Locus Rules ABC

Goal: introduce the first three rules of the Root Locus method.

Reading: FPE, Chapter 5

Reminder: Root Locus



where $L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$, $m \le n$

Root locus: the set of all $s \in \mathbb{C}$ that solve the *characteristic equation*

$$a(s) + Kb(s) = 0$$

as K varies from 0 to ∞.

Six Rules for Sketching Root Loci

There are *six rules* for sketching root loci. These rules are mainly qualitative, and their purpose is to give intuition about impact of poles and zeros on performance.

These rules are:

- Rule A — number of branches
- Rule B — start points
- Rule C — end points
- Rule D — real locus
- Rule E — asymptotes
- Rule F — $j\omega$-crossings

Today, we will cover mostly Rules A–C.

Rule A: Number of Branches

$$1 + K\frac{b(s)}{a(s)} = 1 + K\frac{s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} = 0$$

$$\implies (s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n)$$
$$+ K(s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m) = 0$$

Since $\deg(a) = n \geq m = \deg(b)$, the characteristic polynomial $a(s) + Kb(s) = 0$ has degree n.

The characteristic polynomial has n solutions (roots), some of which may be repeated. As we vary K, these n solutions also vary to form n branches.

Rule A:

$$\#(\text{branches}) = \deg(a)$$

Rule B: Start Points

The locus starts from $K = 0$. What happens near $K = 0$?

If $a(s) + Kb(s) = 0$ and $K \sim 0$, then $a(s) \approx 0$.

Therefore:

- s is close to a root of $a(s) = 0$, or
- s is close to a pole of $L(s)$

Rule B: branches start at open-loop poles.

Rule C: End Points

What happens to the locus as $K \to \infty$?

$$a(s) + Kb(s) = 0$$
$$b(s) = -\frac{1}{K}a(s)$$

— as $K \to \infty$,

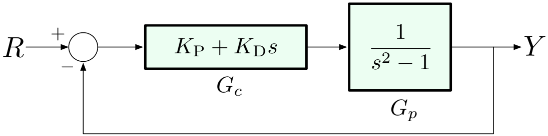
- branches end at the roots of $b(s) = 0$, or
- branches end at zeros of $L(s)$

Rule C: branches end at open-loop zeros.

Note: if $n > m$, we have n branches, but only m zeros. The remaining $n - m$ branches go off to infinity (end at "zeros at infinity").

Example

PD control of an unstable 2nd-order plant



$$\frac{Y}{R} = \frac{G_c G_p}{1 + G_c G_p}$$

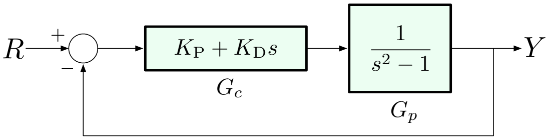
poles: $1 + G_c(s)G_p(s) = 0$

$$1 + (K_\mathrm{P} + K_\mathrm{D}s)\left(\frac{1}{s^2-1}\right) = 0$$

We will examine the impact of varying $K = K_\mathrm{D}$, assuming the ratio $K_\mathrm{P}/K_\mathrm{D}$ *fixed.*

Example

PD control of an unstable 2nd-order plant



We will examine the impact of varying $K = K_D$, assuming the ratio K_P/K_D *fixed*.

Let us write the characteristic equation in *Evans form*:

$$1 + \underbrace{K_D}_{K}\left(s + \frac{K_P}{K_D}\right)\left(\frac{1}{s^2-1}\right) = 1 + K\underbrace{\frac{s + K_P/K_D}{s^2-1}}_{L(s)} = 0$$

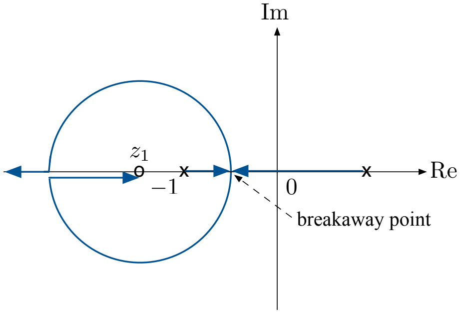
$$L(s) = \frac{s - z_1}{s^2 - 1} \qquad \text{zero at } s = z_1 = -K_P/K_D < 0$$

Example

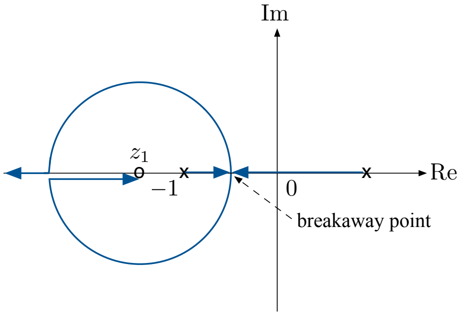
$$L(s) = \frac{s - z_1}{s^2 - 1}$$

- Rule A: $\begin{cases} m = 1 \\ n = 2 \end{cases} \implies$ 2 branches
- Rule B: branches start at open-loop poles $\quad s = \pm 1$
- Rule C: branches end at open-loop zeros $\quad s = z_1, -\infty$
  (we will see why $-\infty$ later)

So the root locus will look something like this:



$$L(s) = \frac{s - z_1}{s^2 - 1}$$



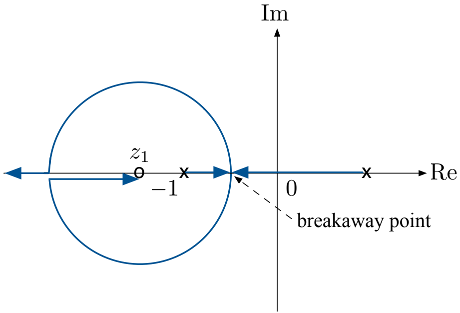
Why does one of the branches go off to $-\infty$?

$s^2 - 1 + K(s - z_1) = 0$

$s^2 + Ks - (Kz_1 + 1) = 0$

$s = -\frac{K}{2} \pm \sqrt{\frac{K^2}{4} + Kz_1 + 1}, \; z_1 < 0$ as $K \to \infty$, s will be < 0

$$L(s) = \frac{s - z_1}{s^2 - 1}$$



Is the point $s = 0$ on the root locus?

Let's see if there is any value $K > 0$, for which this is possible:

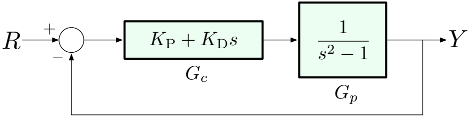
$$1 + KL(0) = 0$$

$$1 + Kz_1 = 0 \qquad K = -\frac{1}{z_1} > 0 \text{ does the job}$$

From Root Locus to Time Response Specs

For concreteness, let's see what happens when

$$K_{\mathrm{P}}/K_{\mathrm{D}} = -z_1 = 2 \qquad \text{and} \qquad K = K_{\mathrm{D}} = 5 \Longrightarrow K_{\mathrm{P}} = 10$$



$$G_c(s) = 10 + 5s$$

$$u = 10e + 5\dot{e}, \qquad e = r - y$$

Characteristic equation: $1 + 5\left(\frac{s+2}{s^2-1}\right) = 0$

$$s^2 + 5s + 9 = 0$$

Relate to 2nd-order response: $\omega_n^2 = 9,\ 2\zeta\omega_n = 5 \Longrightarrow \zeta = 5/6$

Main Points

- When zeros are in LHP, *high gain* can be used to stabilize the system (although one must worry about zeros at infinity).
- If there are zeros in RHP, high gain is always disastrous.
- PD control is effective for stabilization because it introduces a zero in LHP.

But: Rules A–C cannot tell the whole story. How do we know which way the branches go, and which pole corresponds to which zero?

Rules D–F!!