ECE 486: Control Systems

Lecture 14A: Bode Plots for Second-Order Systems

Key Takeaways

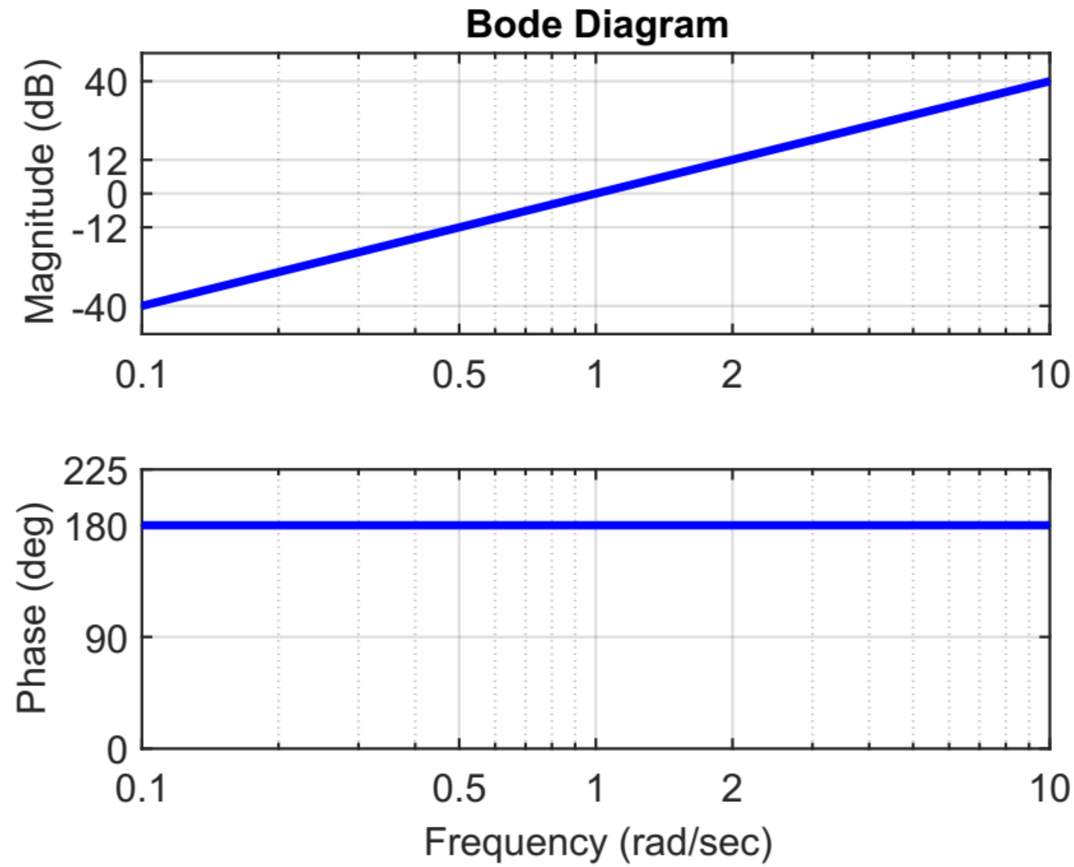
This lecture focuses on Bode plots for second order systems.

- Second-order differentiator $G(s) = s^2$: Phase is +180° and magnitude has slope +40dB/decade.
- Second-order integrator $G(s) = \frac{1}{s^2}$: Phase is -180° and magnitude has slope -40dB/decade:
- Second-order underdamped $G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$:
  - Magnitude is (approximately) flat up to the corner frequency $\omega_n$and rolls off at -40dB/dec at high frequencies.
  - Phase plot transitions by $\pm 180^o$depending on the signs of the coefficients.
  - If damping is low ($\zeta \ll 1$) then the plot has a resonant peak of $|G(j\omega_n)| \approx \left|\frac{G(0)}{2\zeta}\right|$.

Bode Plot: Second-Order Differentiator

- Differentiator: $y(t) = \ddot{u}(t)$ $\quad G(s) = \frac{s^2}{1} = s^2$
  - If $u(t) = \sin(\omega t)$ then $y(t) = -\omega^2 \sin(\omega t) = \omega^2 \sin(\omega t + \pi)$
  - This agrees with $|G(j\omega)| = \omega^2$ and $\angle G(j\omega) = \pi\, rad = +180^o$
- Magnitude has slope +40dB/decade and phase is +180°.

A *N*th order differentiator *G*(*s*) = *s*^{N} has phase +90*Ndeg* and magnitude slope of +20*NdB* per decade.

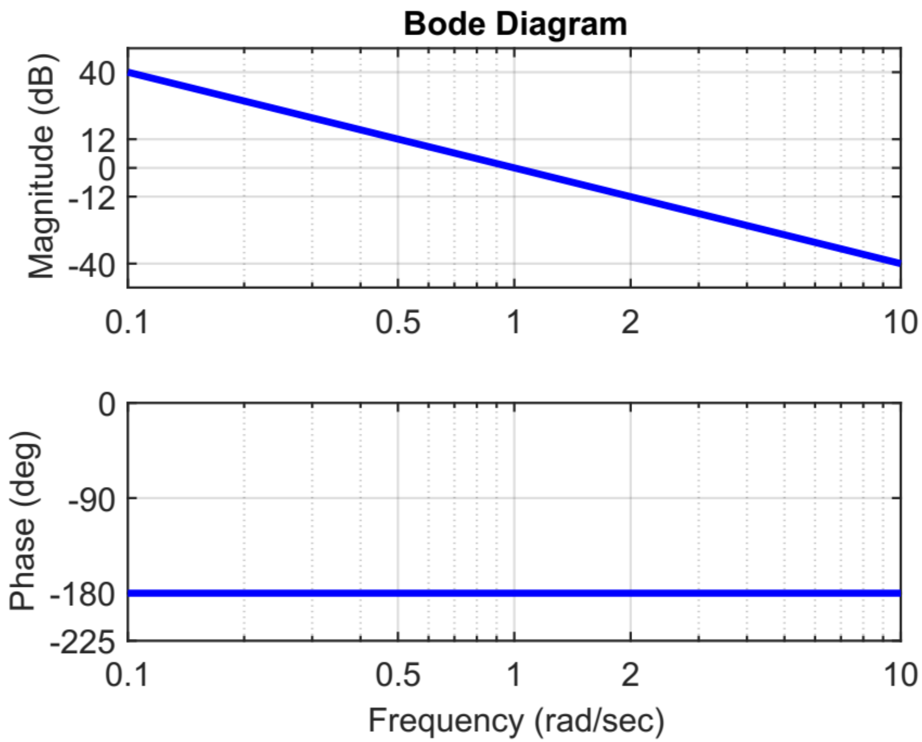


Bode Plot: Second-Order Integrator

- Integrator: $\ddot{y}(t) = u(t)$ $G(s) = \frac{1}{s^2}$
  - If $u(t) = \sin(\omega t)$ then $y(t) = -\frac{1}{\omega^2}\sin(\omega t) = \frac{1}{\omega^2}\sin(\omega t - \pi)$

    [The form for y neglects integration constants.]
  - This agrees with $|G(j\omega)| = \frac{1}{\omega^2}$ and $\angle G(j\omega) = -\pi\, rad = -180^o$
- Magnitude has slope -40dB/decade and phase is -180°.

A N^{th} order integrator $G(s) = \frac{1}{s^N}$ has phase -90*N*deg and magnitude slope of -20*N*dB per decade.



Second-Order Underdamped Systems

Consider the a stable, second-order system:

$$G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$$

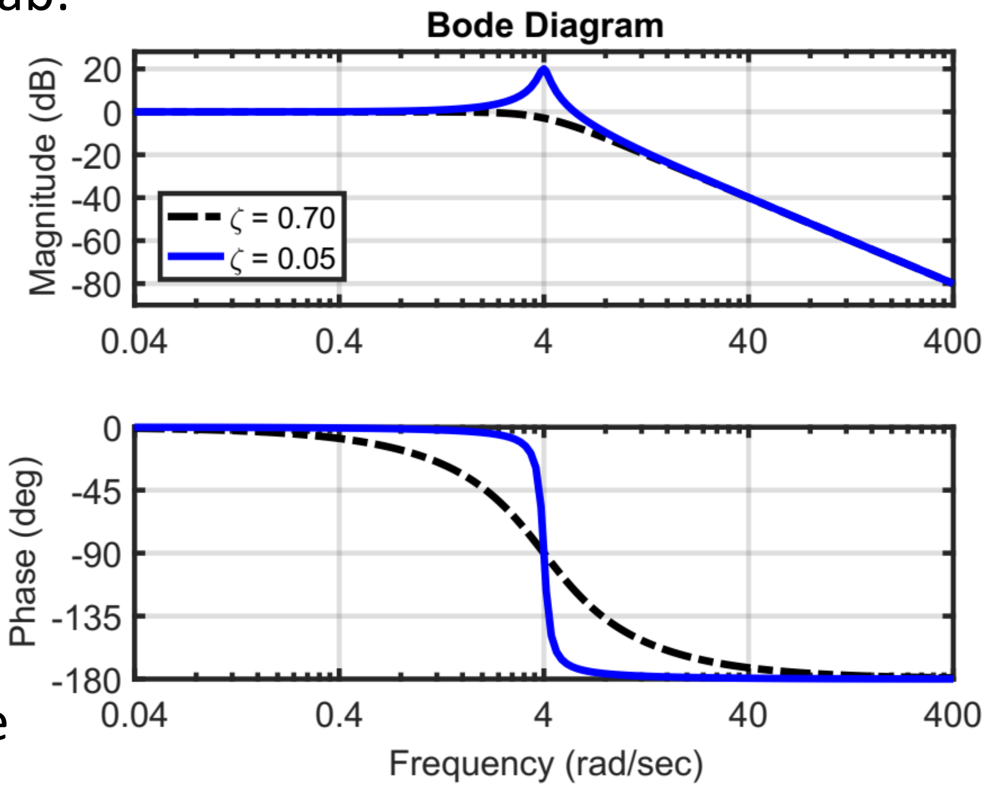
$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = b_0 u(t)$$

Assume $b_0 > 0$.

$$G(s) = \frac{16}{s^2+8\zeta s+16}$$

Bode plots can be generated by Matlab:

```
>> b0=16;
>> wn = 4;
>> z1 = 0.7;
>> G1=tf(b0,[1 2*z1*wn wn^2]);
>> z2 = 0.05;
>> G2=tf(b0,[1 2*z2*wn wn^2]);
>> bode(G1,'k–',G2,'b');
```



It will be useful to sketch straight-line approximate Bode plots.

Corner Frequency

Consider the a stable, second-order system:

$$G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$$

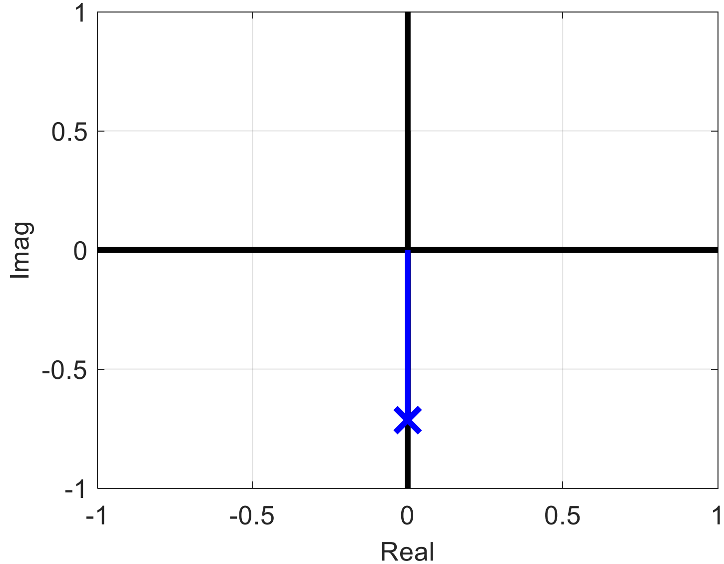
$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = b_0 u(t)$$

Assume $b_0 > 0$.

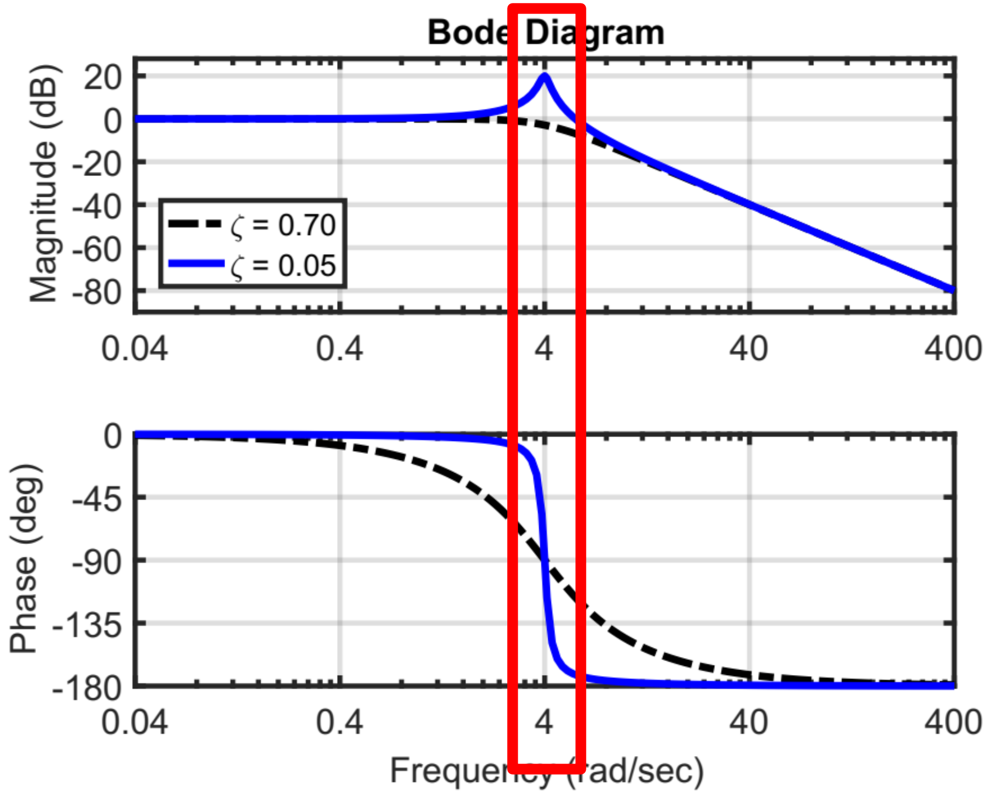
Corner Frequency: $\omega = \omega_n$

$$G(j\omega_n) = \frac{b_0}{(j\omega_n)^2+2\zeta\omega_n(j\omega_n)+\omega_n^2}$$

$$G(j\omega_n) = \frac{b_0}{2\zeta\omega_n^2 j} = -\frac{G(0)}{2\zeta}j$$



$$G(s) = \frac{16}{s^2+8\zeta s+16}$$



Corner Frequency

Consider the a stable, second-order system:

$$G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = b_0 u(t)$$

Assume $b_0 > 0$.

Corner Frequency: $\omega = \omega_n$

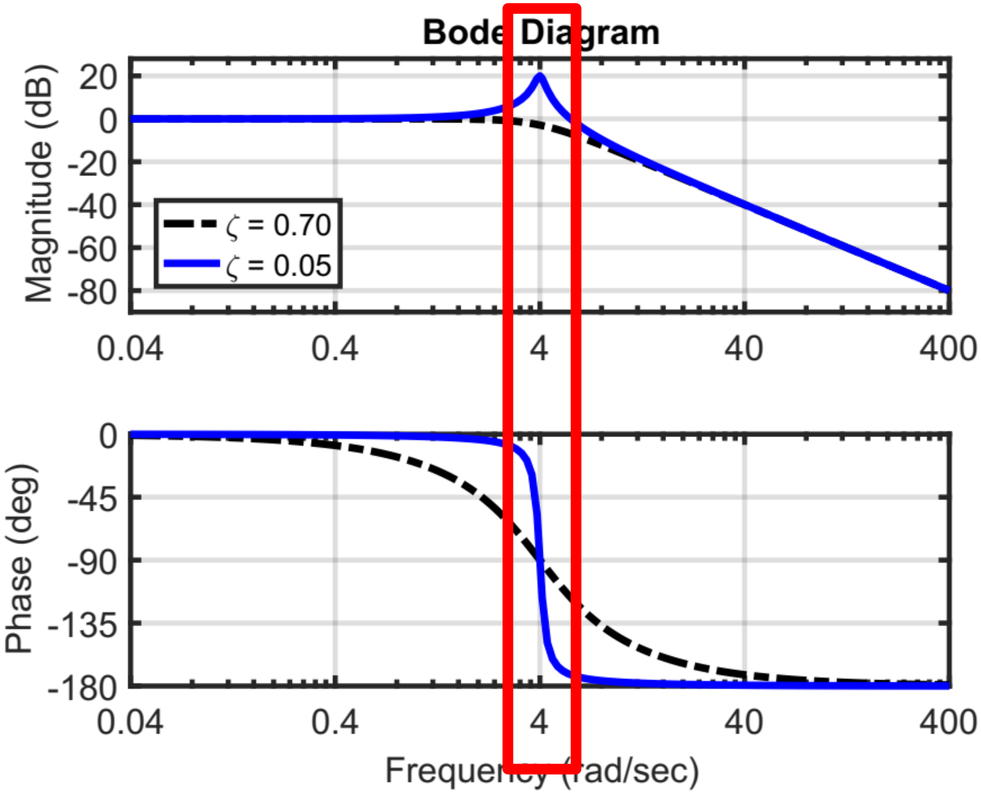
$$G(j\omega_n) = \frac{b_0}{(j\omega_n)^2+2\zeta\omega_n(j\omega_n)+\omega_n^2}$$

$$G(j\omega_n) = \frac{b_0}{2\zeta\omega_n^2 j} = -\frac{G(0)}{2\zeta}j$$

$$\angle G(j\omega_n) = -90^o$$

$$|G(j\omega_n)| = \frac{1}{2\zeta}|G(0)|$$

$$G(s) = \frac{16}{s^2+8\zeta s+16}$$



Small ζ gives a resonant peak.

This is associated with overshoot and oscillations.

Low-Frequency Approximation

Consider the a stable, second-order system:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = b_0 u(t)$$

$$G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Assume $b_0 > 0$.

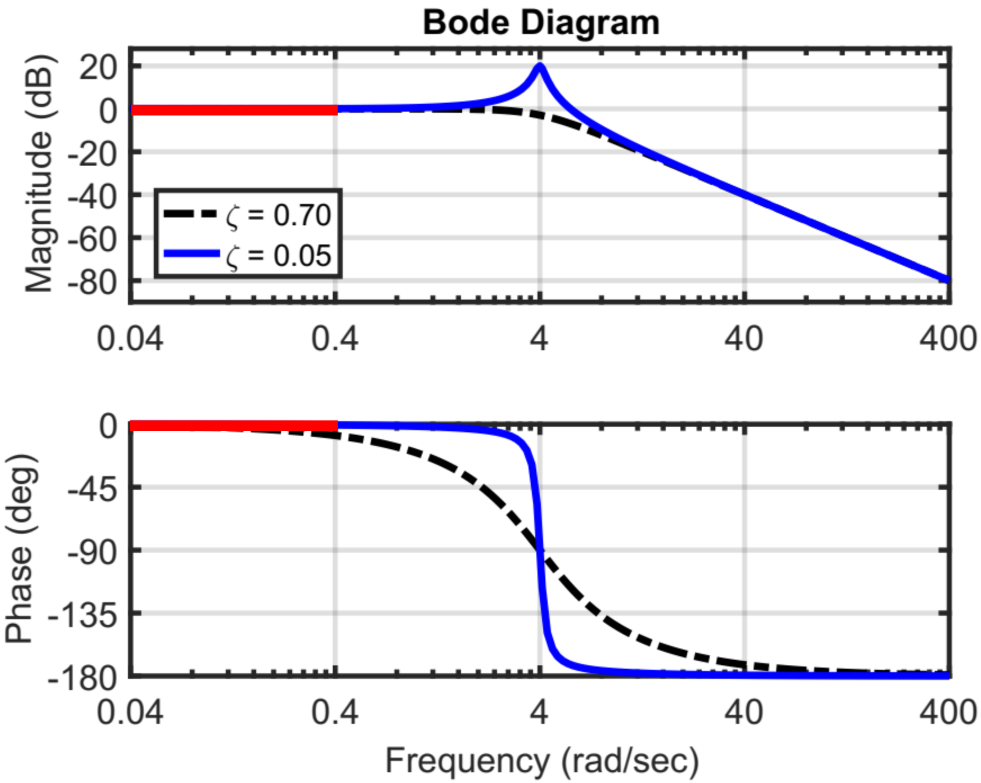
$$G(s) = \frac{16}{s^2+8\zeta s+16}$$

Low Frequency: $\omega \leq \frac{\omega_n}{10}$

$$G(j\omega) \approx \frac{b_0}{\omega_n^2}$$

$$\angle G(j\omega) = 0^o$$

$$|G(j\omega)| = G(0)$$



High-Frequency Approximation

Consider the a stable, second-order system:

$$G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$$

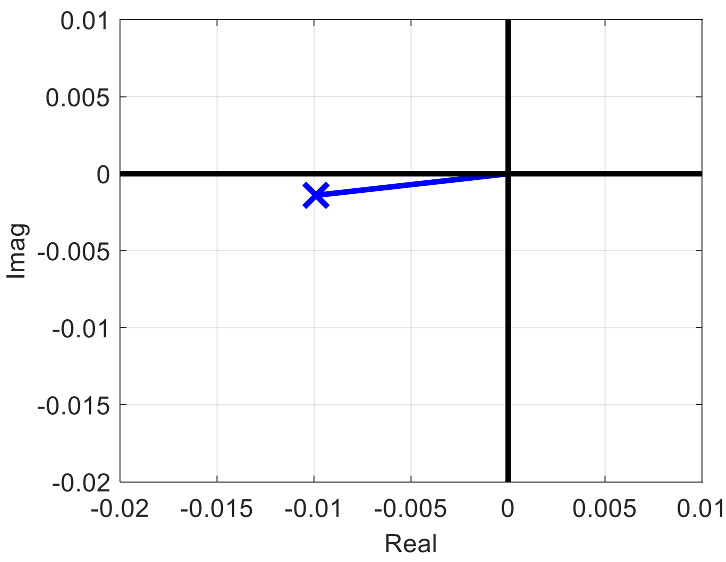
$$\ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_n^2 y(t) = b_0 u(t)$$

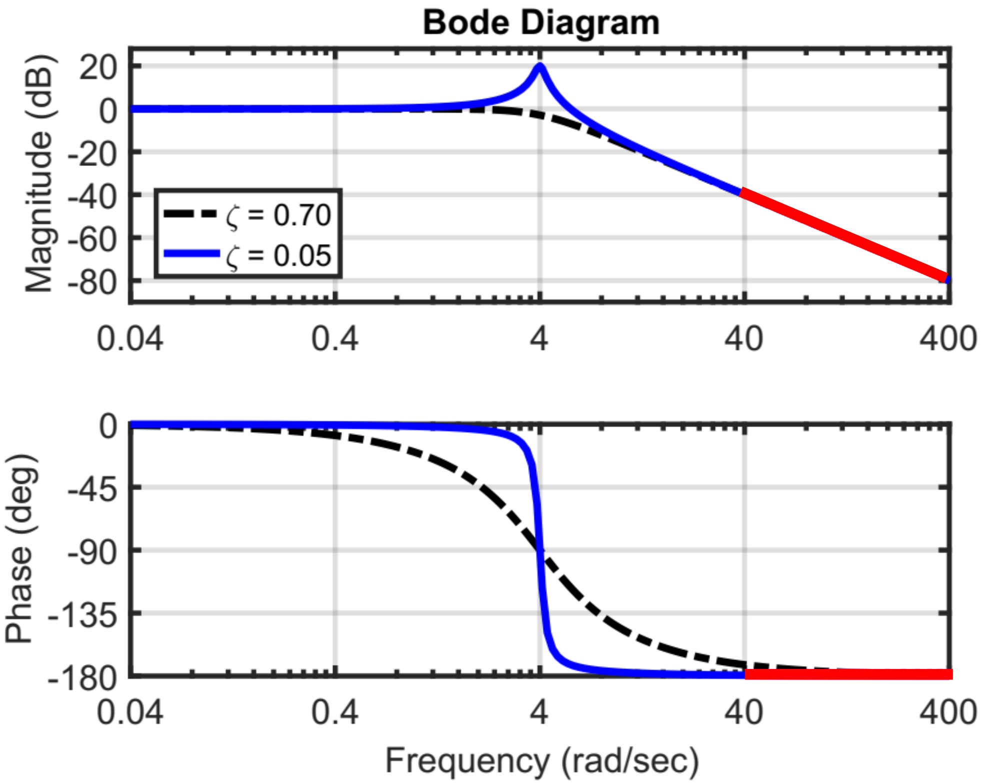
Assume $b_0 > 0$.

$$G(s) = \frac{16}{s^2+8\zeta s+16}$$

High Frequency: $\omega \geq 10\omega_n$

$$G(j\omega) \approx \frac{b_0}{(j\omega)^2} = -\frac{b_0}{\omega^2}$$





High-Frequency Approximation

Consider the a stable, second-order system:

$$G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = b_0 u(t)$$

Assume $b_0 > 0$.

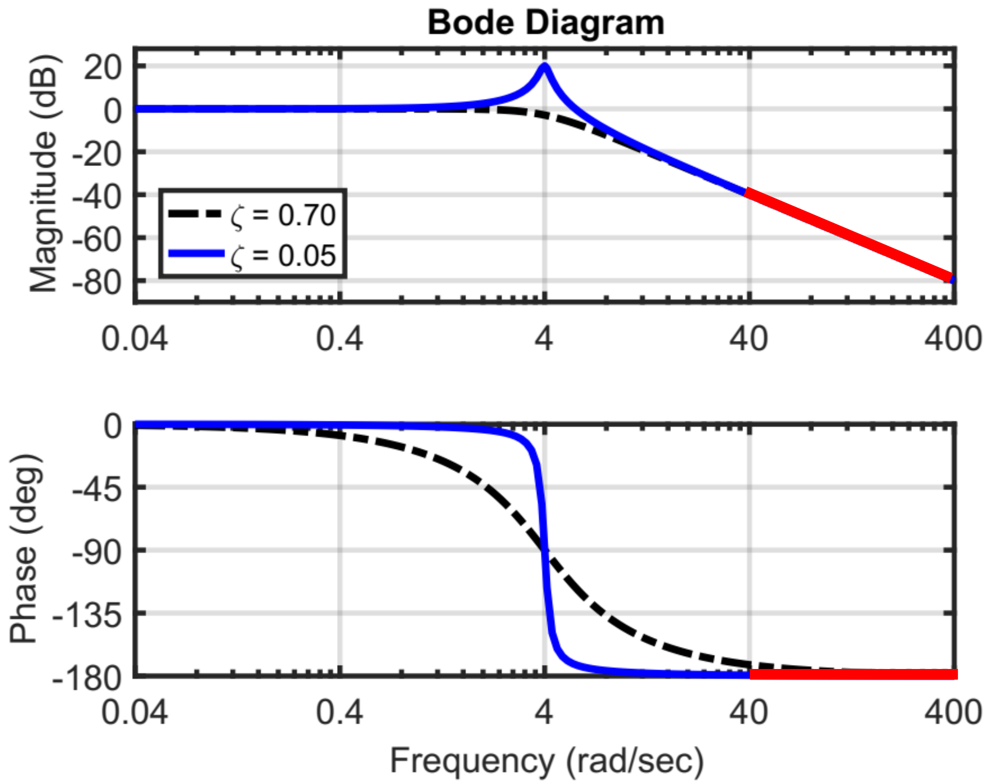
$$G(s) = \frac{16}{s^2+8\zeta s+16}$$

High Frequency: $\omega \geq 10\omega_n$

$$G(j\omega) \approx \frac{b_0}{(j\omega)^2} = -\frac{b_0}{\omega^2}$$

$$\angle G(j\omega) \approx -180^o$$

$$|G(j\omega)| \approx \frac{b_0}{\omega^2}$$



Magnitude rolls-off at -40dB per decade (similar to $1/s^2$).

Middle-Frequency Approximation

Consider the a stable, second-order system:

$$G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = b_0 u(t)$$

Assume $b_0 > 0$.

$$G(s) = \frac{16}{s^2+8\zeta s+16}$$

Middle Frequency:

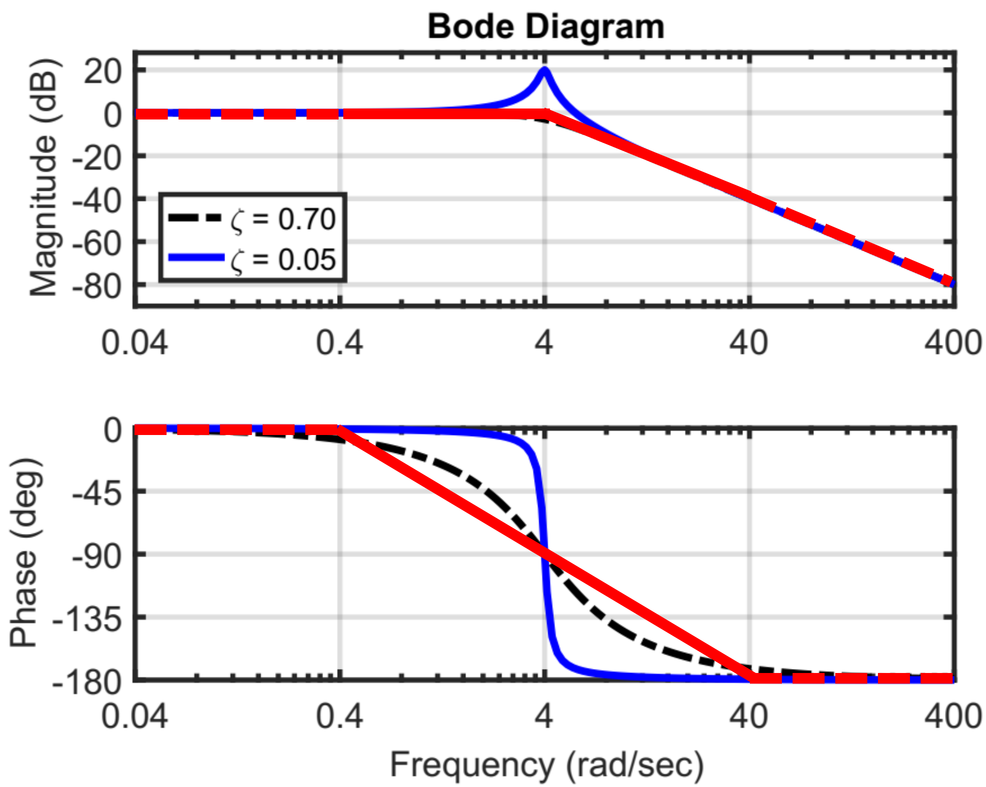
$$\frac{\omega_n}{10} \leq \omega \leq 10\omega_n$$

Straight line approximation to connect low/high freqs.

Magnitude: Lines meet at corner frequency.

Phase: Line passes through -90° at corner frequency.

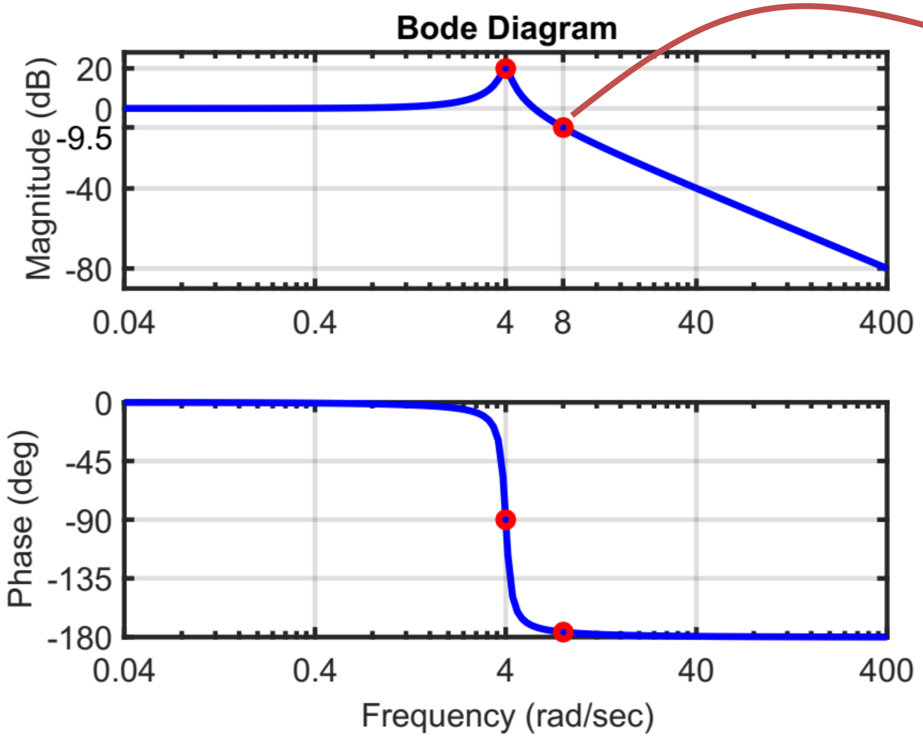
Low ζ gives resonant peak and sharp phase transition.

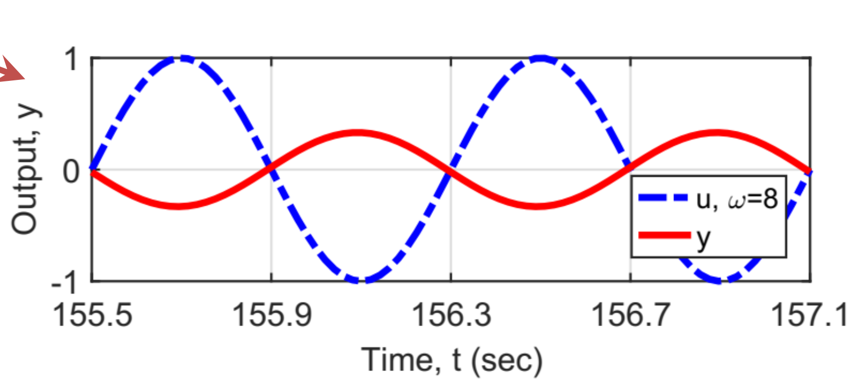


Resonance

- "Lightly" damped second-order system:

$$G(s) = \frac{16}{s^2+8\zeta s+16} \text{ with } \zeta = 0.05$$

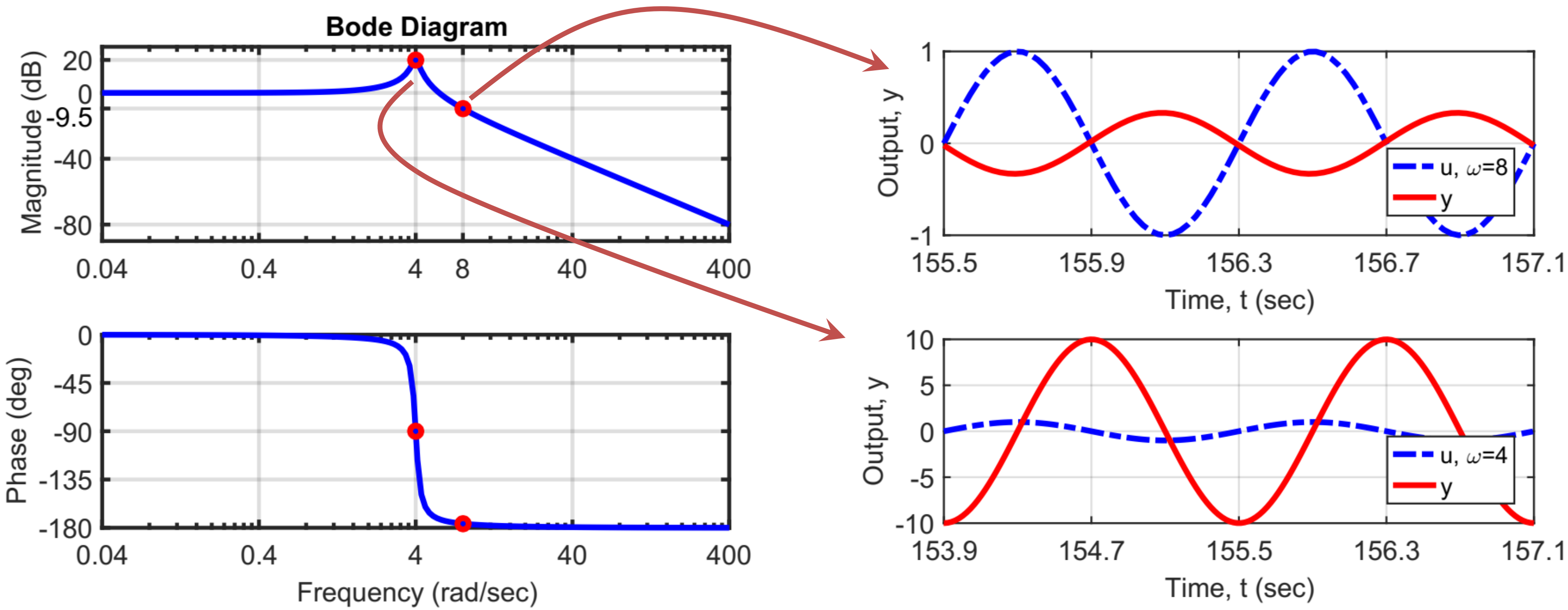




Resonance

- "Lightly" damped second-order system:

$$G(s) = \frac{16}{s^2+8\zeta s+16} \text{ with } \zeta = 0.05$$



General Second-Order Systems

We can draw Bode plots for the following cases using a similar procedure:

$$G(s) = \frac{b_0}{s^2 \pm 2\zeta\omega_n s \pm \omega_n^2} \text{ with } \zeta < 1$$

$$G(s) = \frac{s^2 \pm 2\zeta\omega_n s \pm \omega_n^2}{a_0} \text{ with } \zeta < 1$$

Bode plots for higher-order systems are discussed next. The approach can be used to sketch Bode plots for overdamped second-order systems (which can be expressed as a connection of two first-order systems).