ECE 486: Control Systems

Lecture 16A: Sensitivity Functions

Key Takeaways

This lectures considers a generic feedback system with plant *G(s)* and controller *K(s)*.

Two important transfer functions are:

- Sensitivity: $S(s) = \frac{1}{1+G(s)K(s)}$
- Complementary Sensitivity: $T(s) = \frac{G(s)K(s)}{1+G(s)K(s)}$

The feedback system is defined to be stable if every possible input/output transfer function is internally stable.

- This holds if and only if all zeros of 1+G(s)K(s) are in the LHP.
- The feedback system is unstable if the G(s)K(s) has a pole/zero cancellation in the CRHP.

Problem 1

Consider the feedback system below.

A) What is the transfer function from disturbance *d* to output *y*? Express your answer in terms of *G(s)* and *K(s)*.

B) Is the feedback system stable if $G(s) = \frac{1}{s-2}$ and $K(s) = 5$?

C) Is the feedback system stable if $G(s) = \frac{s-1}{s+2}$ and $K(s) = \frac{5}{s-1}$?

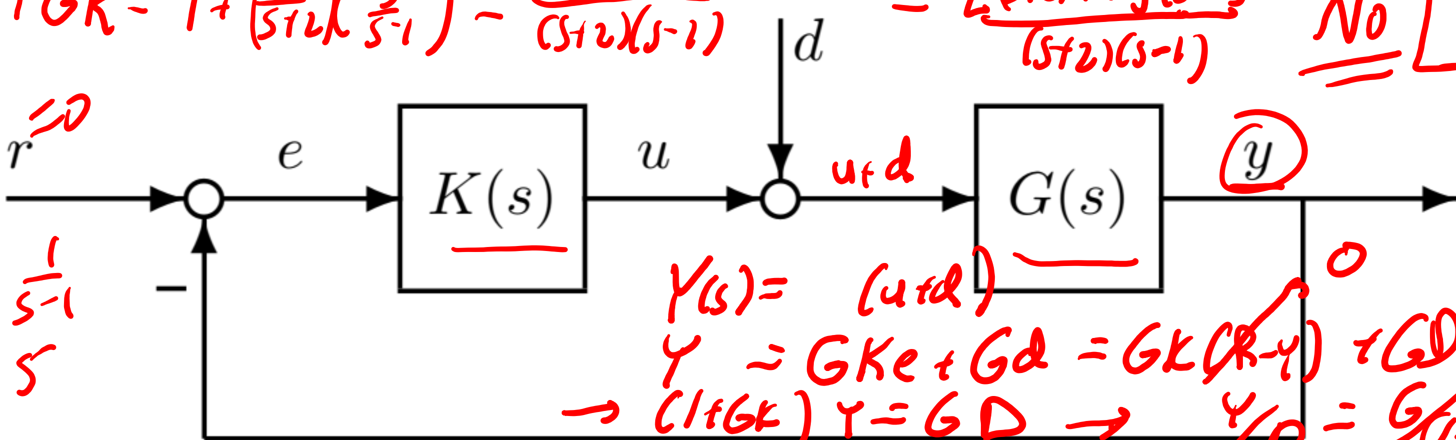
A) $T_{d\to y} = \dfrac{G}{1+GK}$ ✓

B) Yes, $\dfrac{1/(s-2)}{1+(1/(s-2))(5)} = \dfrac{s-2}{(s-2)+5} = \boxed{\dfrac{1}{s+3}}$ ←

Note: $1+GK = 1+\left(\frac{1}{s-2}\right)5 = \dfrac{s+3}{s-2}$ ← zero at $s=-3$

C) $1+GK = 1+\left(\frac{s-1}{s+2}\right)\left(\frac{5}{s-1}\right) = \dfrac{(s+2)(s-1)+5(s-1)}{(s+2)(s-1)} = \dfrac{[(s+2)+5](s-1)}{(s+2)(s-1)}$

No $\boxed{s=+1}$



$G = \frac{1}{s-1}$

$K = 5$

$Y(s) = (u+d)G$

$Y = GKe + Gd = GK(R-Y) + GD$

$\rightarrow (1+GK)Y = GD \rightarrow Y/D = G/(1+GK)$

Solution 1A

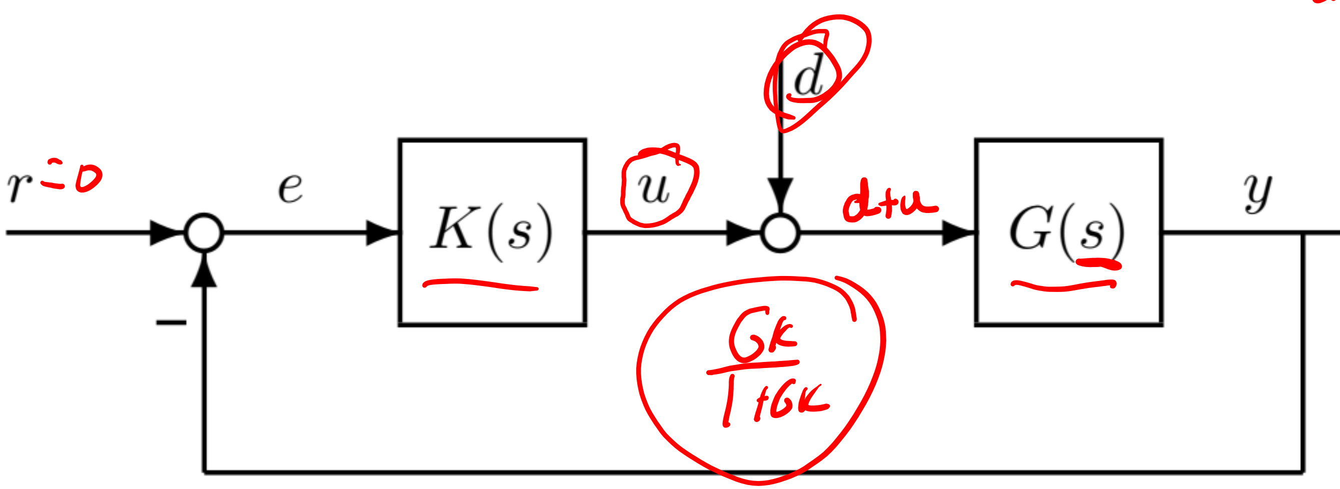
A) What is the transfer function from disturbance *d* to output *y*? Express your answer in terms of *G(s)* and *K(s)*.

$$Y = G(D+u) = G(D+KE) = GD + GK\cdot E$$

$$Y = GD + GK(-Y)$$

$$(1+GK)Y = GD \rightarrow \frac{Y}{D} = \boxed{\frac{G}{1+GK}}$$

$T_{d\to y}$



$$\frac{GK}{1+GK} = \frac{N/D}{1+(N/D)} = \frac{N}{(D+N)}$$

Solution 1B

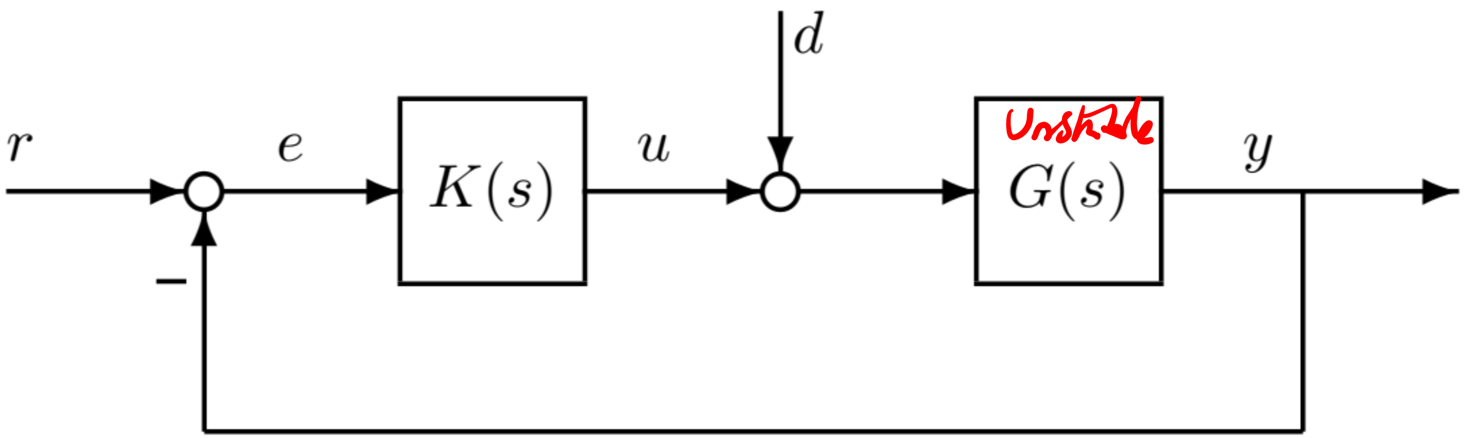
B) Is the feedback system stable if $G(s) = \frac{1}{s-2}$ and $K(s) = 5$?

Closed-loop is Stable if $1 + G(s)K(s)$ has only LHP zeros

$$1 + \left(\frac{1}{s-2}\right)5 = \frac{(s-2)+5}{s-2} = \boxed{\frac{s+3}{s-2}}$$

zero at $s = -3$ in LHP ✓ Stable

$$T_{d \to y} = \frac{G}{\boxed{1+GK}} \qquad \frac{1/s-2}{(s+3)/(s-2)} = \frac{1}{s+3} \quad \boxed{\left(\frac{s-2}{s-2}\right)}$$



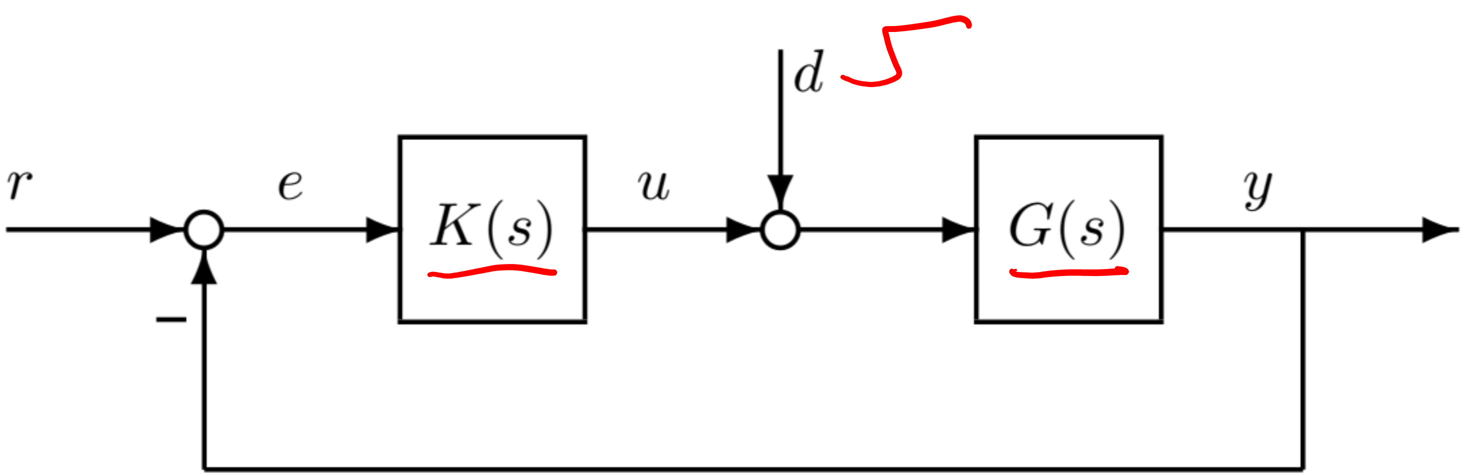
Solution 1C

C) Is the feedback system stable if $G(s) = \frac{s-1}{s+2}$ and $K(s) = \frac{5}{s-1}$?

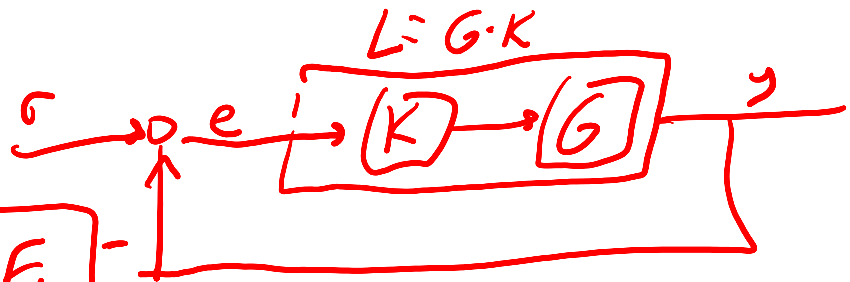
$$1 + GK = 1 + \left(\frac{s-1}{s+2}\right)\left(\frac{5}{s-1}\right) = \frac{(s+2)(s-1) + 5(s-1)}{(s+2)(s-1)}$$

$$= \frac{[(s+2)+5](s-1)}{(s+2)(s-1)} = \frac{(s+7)(s-1)}{(s+2)(s-1)}$$

zero at $s = +1$ in RHP $\Rightarrow$ Unstable Closed loop



Solution 1-Extra Space



$$\boxed{\underset{s+2}{D(s)}\cdot Y = \underset{1}{N(s)}E}$$

$\dot{y} + 2y$ $\qquad$ e

$$D(s)Y = N(s)[R-Y]$$

$$[D+N]Y = NR$$

$$\boxed{\frac{Y}{R} = \frac{N}{N+D}}$$

$$\frac{Y}{R} = \frac{N/D}{(N+D)/D} = \frac{L}{1+L}$$

$$T_{r\to y} = \frac{L(s)}{1+L(s)}$$

$$T_{r\to y} = \frac{\left(\frac{1}{s+2}\right)}{1+\left(\frac{1}{s+2}\right)} = \frac{\frac{1}{s+2}}{\left(\frac{s+3}{s+2}\right)} = \left(\frac{1}{s+3}\right)\left(\frac{s+2}{s+2}\right)$$

$$T_{r\to y}(s) = \frac{(N/D)}{1+(N/D)} = \frac{N/D}{\left[\frac{N+D}{D}\right]} = \left(\frac{N}{N+D}\right)\left(\frac{D}{D}\right)$$

$$\dot{y} + 2y = e$$

$$\boxed{L = \frac{N(s)}{D(s)}} = \frac{1}{s+2}$$

$$\dot{y} + 2y = (r-y)$$

$$\boxed{\dot{y} + 3y = r}$$

$$T_{r\to y}(s) = \frac{1}{s+3}$$

$$\left(0 = N(s) + D(s)\right) / D(s) \Rightarrow 0 = 1 + G(s)K(s) \leftarrow$$

zeros of are CL Poles

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Lecture 16B: Gain Margin

Key Takeaways

This lecture discusses one safety factor called the gain margin to account for model uncertainty.

- The gain margin is the amount of allowable variation in the gain of the plant before the closed-loop becomes unstable.
- As a rule of thumb, the closed-loop should remain stable for gain variations in the range [0.5, 2] (= ±6dB).

CL poles

$S = \frac{1}{1+L(s)}$ $1+L(s)=0$

It is shown that a gain variation $g_0>0$ causes a closed-loop pole at $s = \pm j\omega_0$ if and only if $L(j\omega_0) = -1/g_0$

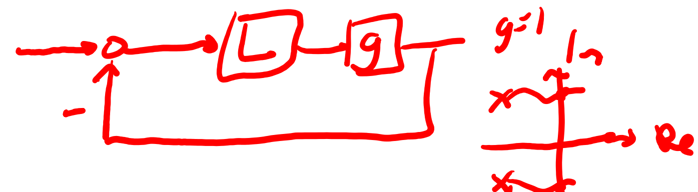
$1 + g\, G(s)\, K(s) = 0$

$1 + g\, G(j\omega)\, K(j\omega) = 0$

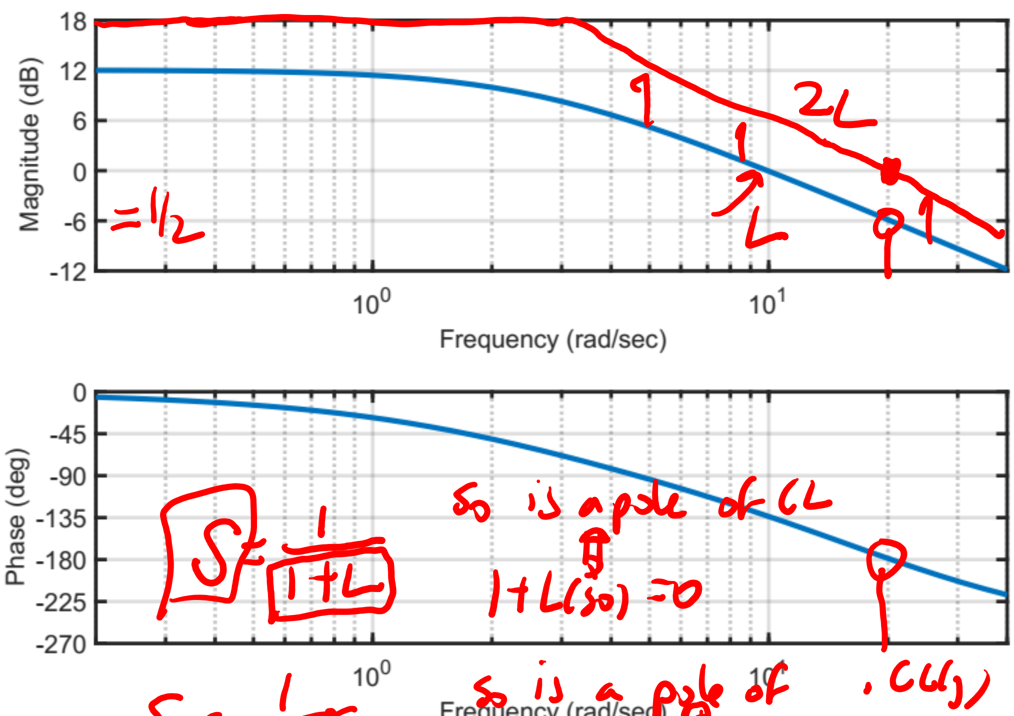
This can be used to determine gain margins from a Bode plot of the loop transfer function *L(s)*.

Problem 2

Consider a standard closed-loop system with the loop transfer function *L(s)* with Bode plot below. Assume the closed-loop is stable with the loop *L(s)*.



$$L(s) = \frac{-4s+72}{0.39s^2+8.02s+18}$$



$=1/2$

$S = \boxed{\frac{1}{1+L}}$

s_0 is a pole of CL

⇕

$1+L(s_0)=0$

$S_g = \frac{1}{1+gL}$

s_0 is a pole of $CL(g)$

⇕

$1+gL(s_0)=0$

$\boxed{1+gL(j\omega)=0}$

A) What is the phase crossover frequency, ω_0? 20 $\boxed{1+gL(j20)=0}$

B) What is the gain margin, g_0, of the closed-loop? $g_0=2$

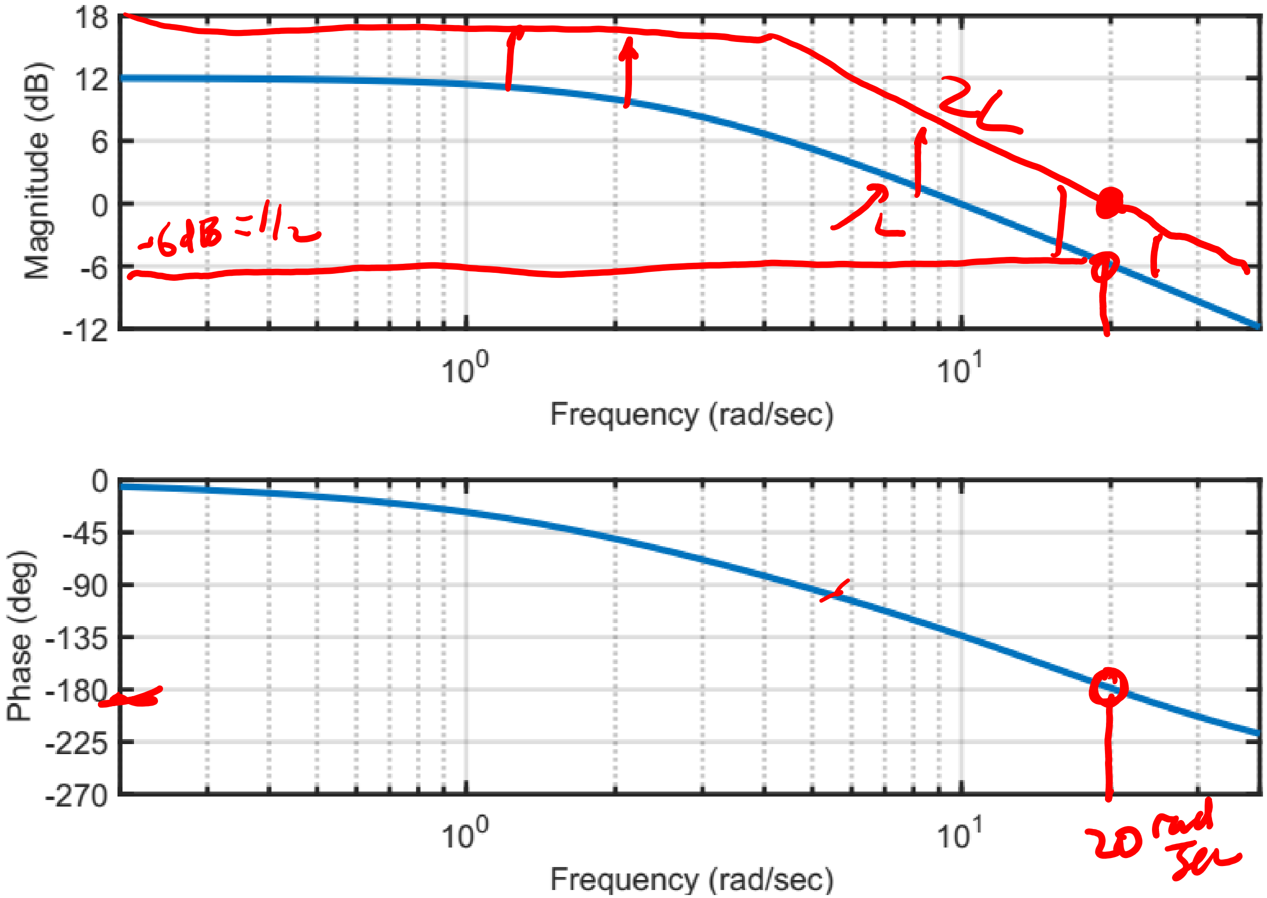
C) Is the closed-loop stable if the open-loop transfer function is *1.5L(s)*? Yes

D) Use Matlab to verify that if the loop is $g_0L(s)$ then the closed-loop has a pole at $j\omega_0$. 0.5, $\boxed{0.7}$, $\boxed{1.4}$, 2.1

Solution 2A and 2B

A) What is the phase crossover frequency, ω_0?

B) What is the gain margin, g_0, of the closed-loop?



$\omega_0 = 20 \frac{rad}{sec}$

$L(j\omega_0) = -1/2$

$$\boxed{1 + \underbrace{g}_{2}\, \underbrace{L(j\omega_0)}_{-1/2} = 0}$$

$gL(j\omega_0) = -1$

$g = 2 = g_0$

$g = 0$

Solution 2C and 2D

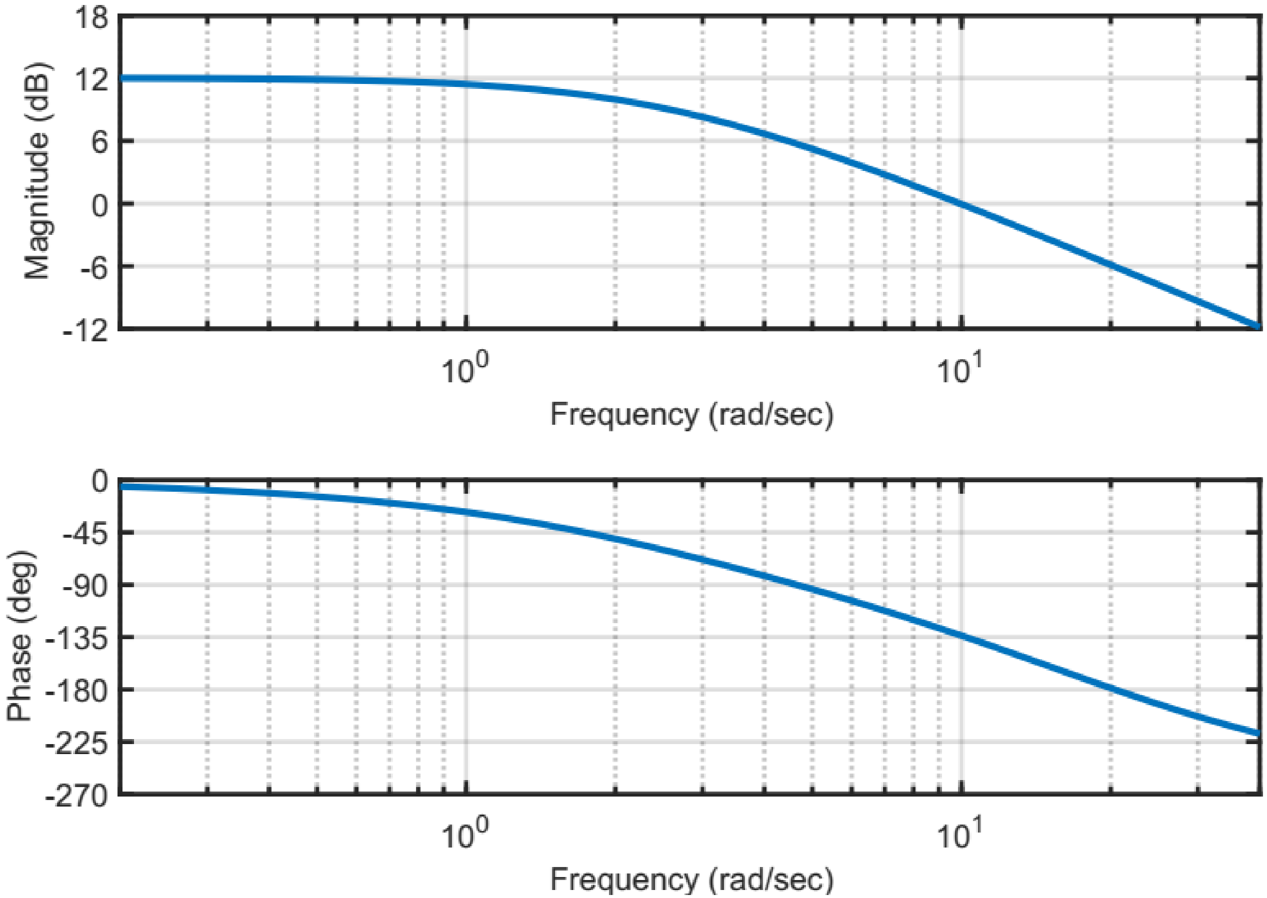
C) Is the closed-loop stable if the open-loop transfer function is 1.5L(s)?

D) Use Matlab to verify that if the loop is $g_0 L(s)$ then the closed-loop has a pole at $j\omega_0$.

$$L(s) = \frac{-4s+72}{0.39s^2+8.02s+18}$$

c) yes

1,5 < g = 2



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Lecture 16C: Phase Margin

Key Takeaways

This lecture discusses another safety factor called the phase margin to account for model uncertainty and time delays.

- The phase margin is the amount of allowable variation in the phase of the plant before the closed-loop becomes unstable.
- As a rule of thumb, the closed-loop should remain stable for phase variations in the range ±45°.

It is shown that a phase variation $\theta_0 > 0$ causes a closed-loop pole at $s = j\omega_0$ if and only if $\boxed{e^{-j\theta_0} L(j\omega_0) = -1.}$

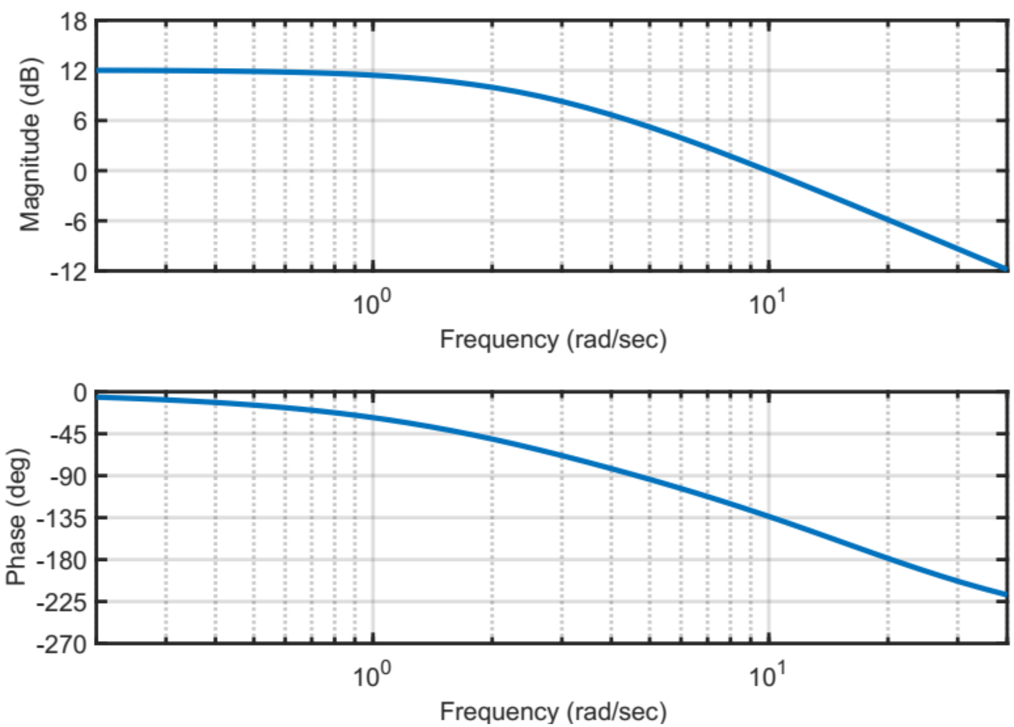
$1 + \underbrace{e^{-j\theta}} L(j\omega) = 0$

This can be used to determine phase margins from a Bode plot of the loop transfer function *L(s)*.

Problem 3

Consider a standard closed-loop system with the loop transfer function *L(s)* with Bode plot below. Assume the closed-loop is stable with the loop *L(s)*.

$$L(s) = \frac{-4s+72}{0.39s^2+8.02s+18}$$



A) What is the gain crossover frequency, ω_0?

B) What is the phase margin, θ_0, of the closed-loop?

C) Use Matlab to verify that if the loop is $e^{-j\theta_0}L(s)$ then the closed-loop has a pole at $j\omega_0$.

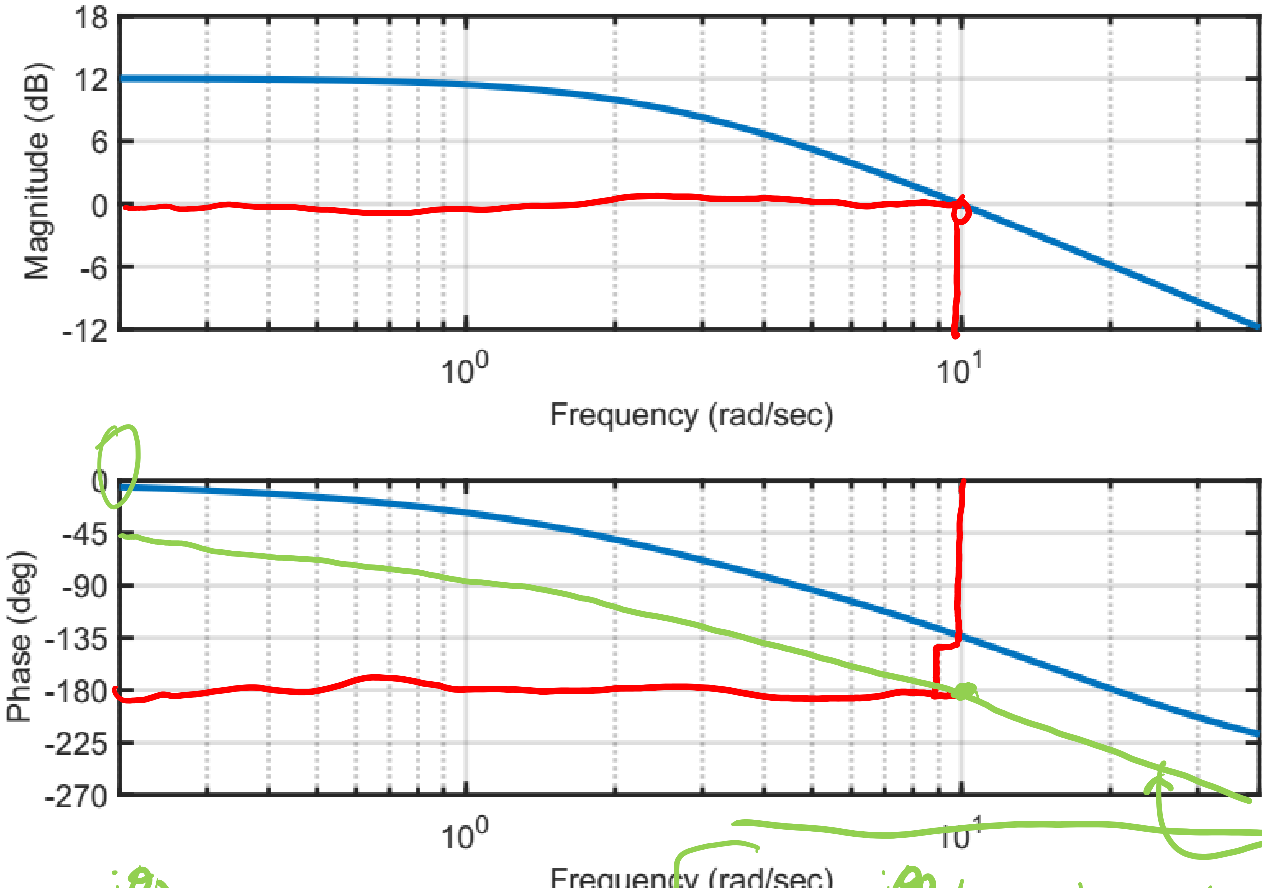
D) What is the delay margin, τ_0, of the closed-loop?

E) Use Matlab to verify (via simulation) that that closed-loop is unstable with a delay of τ_0.

Solution 3A and 3B

A) What is the gain crossover frequency, ω_0?

B) What is the phase margin, θ_0, of the closed-loop?



$\omega_0 = 10\ \text{rad/sec}$

$\angle L(j\omega_0) = -135^\circ$

$\theta_0 = 45^\circ$

$e^{-j\theta_0} \angle$

$\underbrace{\quad}_{|e^{-j\theta_0}|}$

$e^{-j\theta_0} L(j\omega_0) = -1 \longleftrightarrow \boxed{1 + e^{-j\theta_0} L(j\omega_0) = 0}$

Solution 3C

C) Use Matlab to verify that if the loop is $e^{-j\theta_0}L(s)$ then the closed-loop has a pole at $j\omega_0$.