ECE 486: Control Systems

Lecture 4B: First-Order Step Response

Key Takeaways

This lecture covers the step response for first-order systems.

The step response of a *stable*, first-order system.

1. Converges to the final value with neither overshoot nor oscillations.
2. Has a settling time of three time constants.

Solution of First-Order Step Response

Consider the first-order system:

$$\dot{y}(t) + a_0 y(t) = b_0 u(t)$$

with $y(0) = 0$ and $u(t) = 1$ for all $t \geq 0$

$$G(s) = \frac{b_0}{s + a_0}$$

Obtain the response as follows:

1. Solve for roots of the characteristic equation:

$$s + a_0 = 0 \quad \Rightarrow \quad s_1 = -a_0$$

2. Find a particular solution (assume $a_0 \neq 0$):

$$y_P(t) = \bar{y} \quad \Rightarrow \quad a_0 \bar{y} = b_0 \cdot 1 \quad \Rightarrow \quad \bar{y} = \frac{b_0}{a_0} = G(0)$$

3. Form the general solution

$$y(t) = y_P(t) + c_1 e^{s_1 t} = \frac{b_0}{a_0} + c_1 e^{-a_0 t}$$

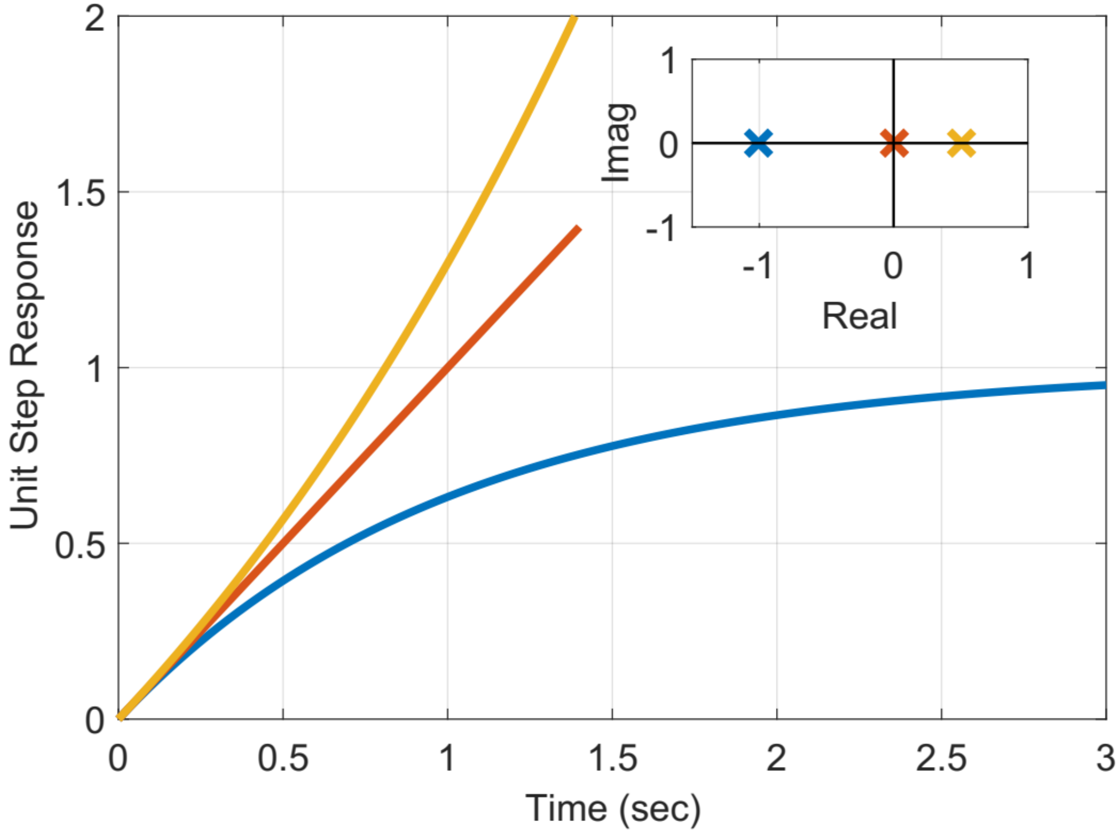
4. Use the initial condition to solve for c_1

$$0 = y(0) = \frac{b_0}{a_0} + c_1 \Rightarrow c_1 = -\frac{b_0}{a_0} \quad \Rightarrow \quad \boxed{y(t) = \frac{b_0}{a_0}\left(1 - e^{-a_0 t}\right)}$$

Stable and Unstable Responses

Response is stable if and only if $s_1 < 0$

$$G(s) = \frac{1}{s-0.5}$$
$$s_1 = +0.5$$
$$y(t) = 1 - e^{0.5t}$$



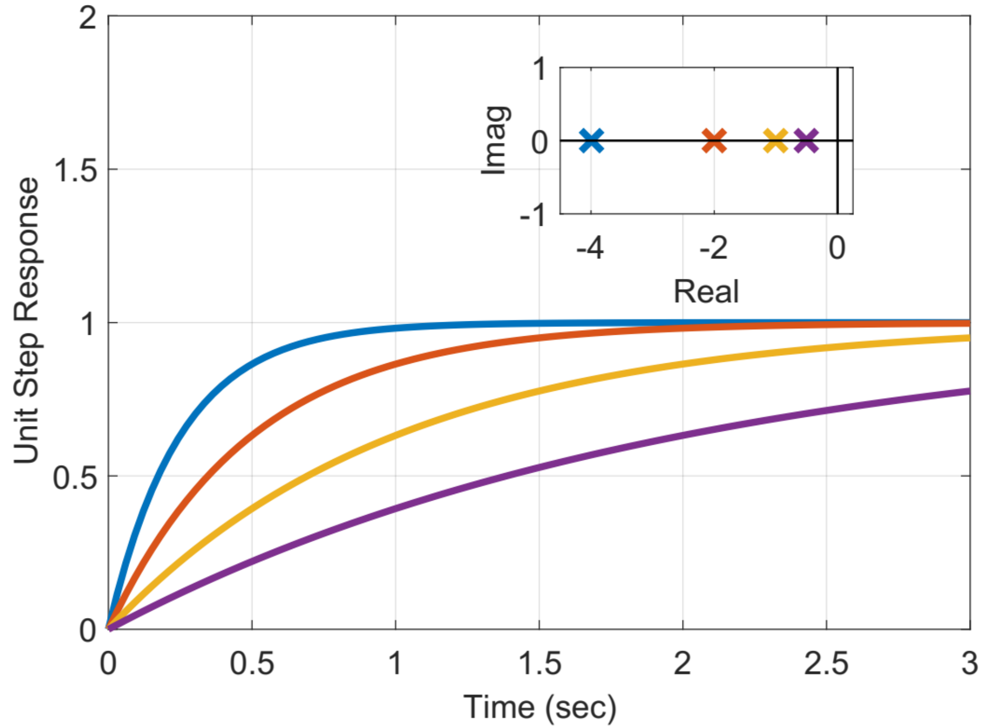
$$G(s) = \frac{1}{s+1}$$
$$s_1 = -1$$
$$y(t) = 1 - e^{-t}$$

Stable Responses

Key features of stable step response:

1. Stable if root is in the LHP: $s_1 < 0$
2. No overshoot
3. Time constant: $\tau = \frac{1}{|s_1|}$
4. Settling time: $3\tau = \frac{3}{|s_1|}$
5. Rise time: roughly 2.2τ
6. Final value:

$$\bar{y} = G(0)\bar{u}$$
$$= G(0) \text{ (if } \bar{u} = 1)$$



Example

Step Response:

$$\dot{y}(t) + 8y(t) = -10u(t) \qquad G(s) = \frac{-10}{s+8}$$

with $y(0) = 0$ and $u(t) = 3$ for all $t \geq 0$

1. Stable: $s + 8 = 0 \quad \Rightarrow \quad s_1 = -8 < 0$

2. Time constant: $\tau = \frac{1}{|s_1|} = \frac{1}{8} = 0.125sec$

3. Settling time: $3\tau = 0.375sec$

4. Final Value: $\bar{y} = G(0)\bar{u} = -\frac{10}{8} \cdot 3 = -3.75$

Example

Step Response:

$$\dot{y}(t) + 8y(t) = -10u(t)$$

$$\text{with } y(0) = 0 \text{ and } u(t) = 3 \text{ for all } t \geq 0$$

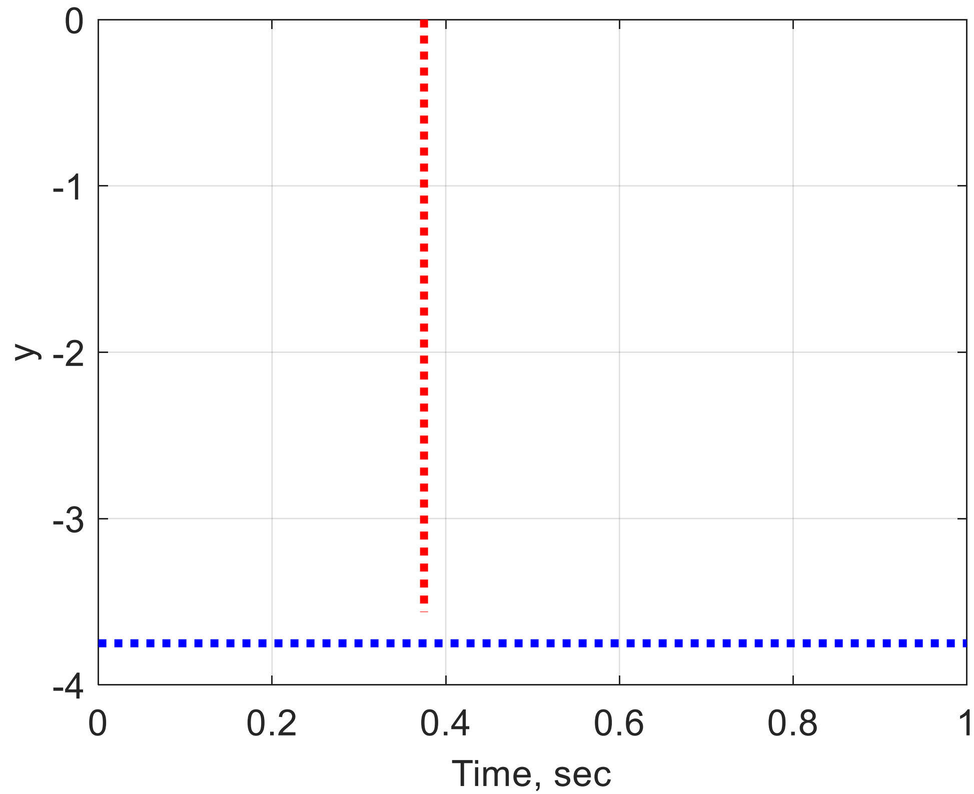
$$G(s) = \frac{-10}{s+8}$$

Response:

(i) stable,

(ii) $3\tau = 0.375$sec,

(iii) $\bar{y}$=-3.75



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Step Response:

$$\dot{y}(t) + 8y(t) = -10u(t)$$

$$G(s) = \frac{-10}{s+8}$$

with $y(0) = 0$ and $u(t) = 3$ for all $t \geq 0$

Response:

(i) stable,

(ii) $3\tau = 0.375$sec,

(iii) $\bar{y}$=-3.75

Matlab:

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>> G=tf(-10,[1 8]);
>> [yunit,t]=step(G);
>> plot(t,3*yunit);
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