ECE 486: Control Systems

Lecture 8A: Proportional-Integral (PI) Control

Key Takeaways

This lecture describes proportional-integral control. The controller sets the plant input with two terms: (i) proportional to the error and (ii) proportional to the integral of the error.

Key properties of PI control:

1. Integral control is that it achieves zero error in steady state (assuming system is stable and reaches a steady-state).
2. Initial transient is dominated by the proportional term while the steady state is dominated by the integral term.

The two terms in a PI controller can be used to provide better trade-offs in the control design.

Problem 1

Consider the following plant:

$$2\dot{y}(t) + 6y(t) = 8u(t)$$

with a PI controller in the following form:

$$u(t) = K_p\, e(t) + K_i \int_0^t e(\tau)\, d\tau$$

A) What is the ODE model for the closed-loop from *r* to *y*?

B) Choose (K_p,K_i) so that the closed-loop: i) is stable and ii) has ζ $=0.5$, iii) settling time $T_s \leq 3sec$, and (iv) zero steady-state error due to a unit step reference.

C) How will the response change if K_i is increased further? Would you recommend increasing K_i based on your analysis?

Solution 1A

$$2\dot{y}(t) + 6y(t) = 8u(t)$$

$$u(t) = K_p\, e(t) + K_i \int_0^t e(\tau)\, d\tau$$

A) What is the ODE model for the closed-loop from *r* to *y*?

$$2\dot{y} + 6y = 8K_p e + 8K_i \int_0^t e$$

$$d/dt$$

$$2\ddot{y} + 6\dot{y} = 8K_p \dot{e} + 8K_i e$$

$$e = r - y$$

$$2\ddot{y} + (6 + 8K_p)\dot{y} + 8K_i y = 8K_p \dot{r} + 8K_i r$$

$$T_{r \to y}(s) = \frac{8K_p s + 8K_i}{2s^2 + (6 + 8K_p)s + 8K_i}$$

Solution 1B

$$2\dot{y}(t) + 6y(t) = 8u(t)$$
$$u(t) = K_p\, e(t) + K_i \int_0^t e(\tau)\, d\tau$$

B) Choose (K_p,K_i) so that the closed-loop: i) is stable and ii) has ζ =0.5, iii) settling time $T_s \leq 3sec$, and (iv) zero steady-state error due to a unit step reference. $\Leftarrow$ Integral Control ensures (iv)

$$T(s) = \frac{8K_p s + 8K_i}{2s^2 + \underbrace{(6+8K_p)}_{>0}s + \underbrace{(8K_i)}_{>0}}$$

$$s^2 + \underbrace{\left(\frac{6+8K_p}{2}\right)}_{>0}s + \underbrace{\left(\frac{8K_i}{2}\right)}_{>0} = 0$$

$$4 = \omega_n^2 = \frac{8K_i}{2} \rightarrow \boxed{K_i = \frac{2\cdot 4}{8} = 1}$$

$$2 = 2\zeta\omega_n = \frac{6+8K_p}{2} \rightarrow 6 + 8K_p = 4 \rightarrow 8K_p = -2 \rightarrow \boxed{K_p = -1/4}$$

Aside:
If $\zeta < 1$
real part
$-\zeta\omega_n$

$3 = T_s = 3/\zeta\omega_n$
$\Rightarrow \zeta\omega_n = 1$
$\zeta = \frac{1}{2}$ $\Rightarrow \omega_n = 2$

Solution 1C

$J\dot{\omega} + b\omega = Ku$

$m\dot{v} + bv = Ku$

$$2\dot{y}(t) + 6y(t) = 8u(t)$$

$$u(t) = K_p\, e(t) + K_i \int_0^t e(\tau)\, d\tau$$

If you keep K_p fixed then

C) ~~How will the response change if~~ K_i is increased further? Would you recommend increasing K_i based on your analysis?

$$\boxed{\begin{aligned} \omega_n^2 &= \frac{8K_i}{2} \\ 2\zeta\omega_n &= \frac{6+8K_p}{2} \end{aligned}}$$

← Unchanged

Picked (K_p, K_i) so that $\zeta = 1/2$, $\omega_n = 2$

Keep K_p the same → $\underline{2\zeta\omega_n}$ stays the same ←

$\underline{\uparrow K_i}$ $\underline{\uparrow \omega_n}$ $\downarrow \zeta$

↑ Overshoot, oscillations (Not good a thing)

↓ Rise Time ← Good

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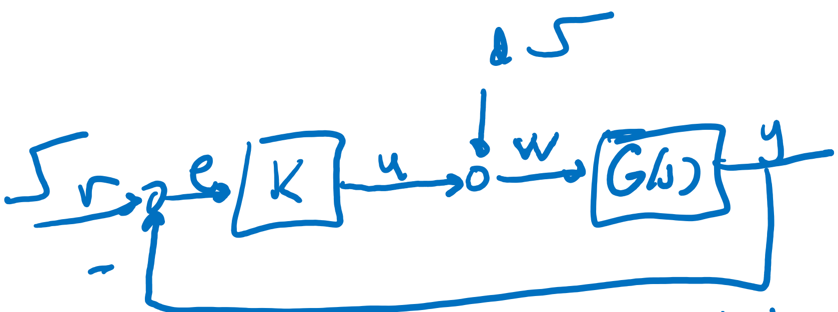
$$\begin{pmatrix} Y \\ U \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} R \\ D \end{pmatrix}$$

$$Y = \begin{bmatrix} T_{r\to y}(s) & T_{d\to y}(s) \end{bmatrix} \begin{bmatrix} R(s) \\ D(s) \end{bmatrix}$$

Lecture 8B: Proportional-Derivative (PD) Control



$$Y = G\cdot U + G\cdot D$$
$$= G(U+D)$$

$$\boxed{\dot y + a_0 y = b_0 w = b_0 u + b_0 d}$$

$$u = K_p(r-y)$$

$$\boxed{\dot y + (a_0 + b_0 K_p) y = (b_0 K_p) r + b_0 d}$$

$$T_{r\to y} = \frac{b_0 K_p}{s + (a_0 + b_0 K_p)}$$

$$T_{d\to y} = \frac{b_0}{s + (a_0 + b_0 K_p)}$$

Key Takeaways

This lecture describes proportional-derivative control. The controller sets the plant input with two terms: (i) proportional to the error and (ii) proportional to the derivative of the error.

Key properties of PD control:

1. Some plants cannot be stabilized by P or PI control. This motivates the use of PD.
2. A basic implementation of PD control will amplify noise.
3. Common implementations use a "smoothed" derivative or a direct measurement the derivative of the output.

Problem 2

Consider the following plant:

$$\ddot{y}(t) - 2\dot{y}(t) + y(t) = u(t)$$

with a PD controller in the following form:

$$u(t) = K_p \underbrace{\left(r(t) - y(t)\right)}_{e} - \underbrace{K_d \dot{y}(t)}_{\text{Rate Feedback}}$$

A) What is the ODE model for the closed-loop from *r* to *y*?

B) Choose (K_p,K_d) so that the closed-loop: i) is stable and ii) has (ω_n ,ζ)= (2 rad/sec,0.5).

C) What is the steady-state error if *r* is a unit step reference?

D) Would you increase or decrease Kp to reduce the steady-state error? What other impacts does this change have on the response?

Solution 2A

$$\ddot{y}(t) - 2\dot{y}(t) + y(t) = u(t)$$

$$u(t) = K_p\left(r(t) - y(t)\right) - K_d\dot{y}(t)$$

A) What is the ODE model for the closed-loop from *r* to *y*?

$$\ddot{y} - 2\dot{y} + y = K_p(r - y) - K_d\dot{y}$$

$$\boxed{\ddot{y} + (K_d - 2)\dot{y} + (K_p + 1)y = K_p r}$$

$$T_{r \to y}(s) = \frac{K_p}{s^2 + (K_d - 2)s + (K_p + 1)}$$

Solution 2B

$$\ddot{y}(t) - 2\dot{y}(t) + y(t) = u(t)$$

$$u(t) = K_p\left(r(t) - y(t)\right) - K_d\dot{y}(t)$$

B) Choose (K_p,K_d) so that the closed-loop: i) is stable and ii) has (ω_n ,ζ)= (2 rad/sec,0.5).

$\ddot{y} - 2\dot{y} + y = K_p(r-y) - K_d\dot{y} \rightarrow \boxed{\ddot{y} + (K_d-2)\dot{y} + (K_p+1)y = K_p r}$

$$s^2 + \underbrace{(K_d-2)}_{2\zeta\omega_n} s + \underbrace{(K_p+1)}_{\omega_n^2} = 0$$

$4 = \omega_n^2 = K_p + 1 \rightarrow K_p = 3$

$2 = 2\zeta\omega_n = K_d - 2 \rightarrow K_d = 4$

$$\Rightarrow \boxed{\ddot{y} + 2\dot{y} + 4y = 3r}$$

Solution 2C

$$\ddot{y}(t) - 2\dot{y}(t) + y(t) = u(t)$$

$$u(t) = K_p\left(r(t) - y(t)\right) - K_d\dot{y}(t)$$

C) What is the steady-state error if r is a unit step reference?

$$0 \leftarrow \boxed{\ddot{y} + 2\dot{y}} + 4y = 3r$$

If $r = \bar{r}$ then $y \to \bar{y}$

$$4\bar{y} = 3\bar{r} \rightarrow \bar{y} = \frac{3}{4}\bar{r} = \frac{3}{4}$$

$$\bar{r} = 1$$

$$T_{r\to y}(0) = \frac{3}{4}$$

$$\boxed{\bar{e} = \bar{r} - \bar{y} = 1 - 3/4 = 1/4}$$

Solution 2D

$$\ddot{y}(t) - 2\dot{y}(t) + y(t) = u(t)$$

$$u(t) = K_p\left(r(t) - y(t)\right) - K_d\dot{y}(t)$$

D) Would you increase or decrease Kp to reduce the steady-state error? What other impacts does this change have on the response?

$$0 \leftarrow \boxed{\ddot{y} + (K_d - 2)\dot{y}} + \underbrace{(K_p+1)}_{=3} y = K_p r$$

$$(K_p+1)\bar{y} = K_p \bar{r} \rightarrow \bar{y} = \left(\frac{K_p}{K_p+1}\right)\underbrace{\bar{r}}_{=1} = \frac{K_p}{K_p+1}$$

$$\uparrow K_p \quad \bar{y} \rightarrow \bar{r} = 1 \quad \downarrow \bar{e}$$

(This will have other consequences, e.g. $\uparrow u$, $\downarrow P$)