

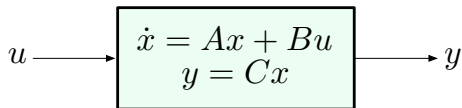
Plan of the Lecture

- ▶ **Review:** state-space notions: canonical forms, controllability.
- ▶ **Today's topic:** controllability, stability, and pole-zero cancellations; effect of coordinate transformations; conversion of any controllable system to CCF.

Goal: explore the effect of pole-zero cancellations on internal stability; understand the effect of coordinate transformations on the properties of a given state-space model (transfer function; open-loop poles; controllability).

Reading: FPE, Chapter 7

State-Space Realizations



- ▶ a given transfer function $G(s)$ can be realized using infinitely many state-space models
- ▶ certain properties make some realizations preferable to others
- ▶ one such property is *controllability*

Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \quad y = Cx \quad x \in \mathbb{R}^n$$

The **Controllability Matrix** is defined as

$$\mathcal{C}(A, B) = [B \mid AB \mid A^2B \mid \dots \mid A^{n-1}B]$$

We say that the above system is **controllable** if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

- ▶ As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- ▶ Whether or not the system is controllable depends on its state-space realization.

Example: Computing $\mathcal{C}(A, B)$

Let's get back to our old friend:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_B u, \quad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2 \times 2} \implies \mathcal{C}(A, B) \in \mathbb{R}^{2 \times 2}$

$$\begin{aligned} \mathcal{C}(A, B) &= [B \mid AB] & AB &= \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \\ \implies \mathcal{C}(A, B) &= \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix} \end{aligned}$$

Is this system controllable?

$$\det \mathcal{C} = -1 \neq 0 \quad \implies \quad \text{system is controllable}$$

Controller Canonical Form

A single-input state-space model

$$\dot{x} = Ax + Bu, \quad y = Cx$$

is said to be in **Controller Canonical Form** (CCF) if the matrices A, B are of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A system in CCF is *always controllable!!*

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

CCF with Arbitrary Zeros

In our example, we had $G(s) = \frac{s+1}{s^2+5s+6}$, with a minimum-phase zero at $z = -1$.

Let's consider a general zero location $s = z$:

$$G(s) = \frac{s-z}{s^2+5s+6}$$

This gives us a CCF realization

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_B u, \quad y = \underbrace{\begin{pmatrix} -z & 1 \end{pmatrix}}_C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since A, B are the same, $C(A, B)$ is the same \implies the system is still controllable.

A system in CCF is controllable for any locations of the zeros.

OCF with Arbitrary Zeros

Start with the CCF

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_B u, \quad y = \underbrace{\begin{pmatrix} -z & 1 \end{pmatrix}}_C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Convert to OCF: $(A \mapsto A^T, B \mapsto C^T, C \mapsto B^T)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}}_{\bar{A}=A^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} -z \\ 1 \end{pmatrix}}_{\bar{B}=C^T} u, \quad y = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{\bar{C}=B^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We already know that this system realizes the same t.f. as the original system.

But is it *controllable*?

OCF with Arbitrary Zeros

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}}_{\bar{A}=A^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} -z \\ 1 \end{pmatrix}}_{\bar{B}=C^T} u, \quad y = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{\bar{C}=B^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let's find the controllability matrix:

$$\mathcal{C}(\bar{A}, \bar{B}) = [\bar{B} \mid \bar{A}\bar{B}] \quad \bar{A}\bar{B} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} -z \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -z-5 \end{pmatrix}$$

$$\therefore \mathcal{C}(\bar{A}, \bar{B}) = \begin{pmatrix} -z & -6 \\ 1 & -z-5 \end{pmatrix}$$

$$\det \mathcal{C} = z(z+5) + 6 = z^2 + 5z + 6 = 0 \quad \text{for } z = -2 \text{ or } z = -3$$

The OCF realization of the transfer function

$G(s) = \frac{s-z}{s^2+5s+6}$ is not controllable when $z = -2$ or -3 , even though the CCF is always controllable.

Beware of Pole-Zero Cancellations!

The OCF realization of the transfer function

$$G(s) = \frac{s - z}{s^2 + 5s + 6}$$

is not controllable when $z = -2$ or -3 , even though the CCF is always controllable.

Let's examine $G(s)$ when $z = -2$:

$$G(s) = \frac{s - z}{s^2 + 5s + 6} \Big|_{z=-2} = \frac{\cancel{s+2}}{(\cancel{s+2})(s+3)} = \frac{1}{s+3}$$

— pole-zero cancellation!

For $z = -2$, $G(s)$ is a first-order transfer function, which can always be realized by this 1st-order controllable model:

$$\dot{x}_1 = -3x_1 + u, \quad y = x_1 \quad \longrightarrow \quad G(s) = \frac{1}{s+3}$$

Beware of Pole-Zero Cancellations!!

We can look at this from another angle: consider the t.f.

$$G(s) = \frac{1}{s + 3}$$

We can realize it using a one-dimensional controllable state-space model

$$\dot{x}_1 = -3x_1 + u, \quad y = x_1$$

or a noncontrollable two-dimensional state-space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u, \quad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

— certainly not the best way to realize a simple t.f.!

Thus, even the *state dimension* of a realization of a given t.f. is not unique!!

Beware of Pole-Zero Cancellations!!

Here is a **really bad** realization of the t.f.

$$G(s) = \frac{1}{s + 3}.$$

Use a two-dimensional model:

$$\dot{x}_1 = -3x_1 + u$$

$$\dot{x}_2 = 100x_2$$

$$y = x_1$$

- ▶ x_2 is not affected by the input u (i.e., it is an uncontrollable mode), and not visible from the output y
- ▶ does not change the transfer function
- ▶ ... and yet, horrible to implement: $x_2(t) \propto e^{100t}$

The transfer function can mask undesirable internal state behavior!!

Pole-Zero Cancellations and Stability

- ▶ In case of a pole-zero cancellation, the t.f. contains *much less* information than the state-space model because some dynamics are “hidden.”
- ▶ These dynamics can be either good (stable) or bad (unstable), but we cannot tell from the t.f.
- ▶ Our original definition of stability (no RHP poles) is flawed because there can be RHP eigenvalues of the system matrix A that are canceled by zeros, yet they still have dynamics associated with them.

Definition of Internal Stability (State-Space Version): a state-space model with matrices (A, B, C, D) is *internally stable* if all eigenvalues of the A matrix are in LHP.

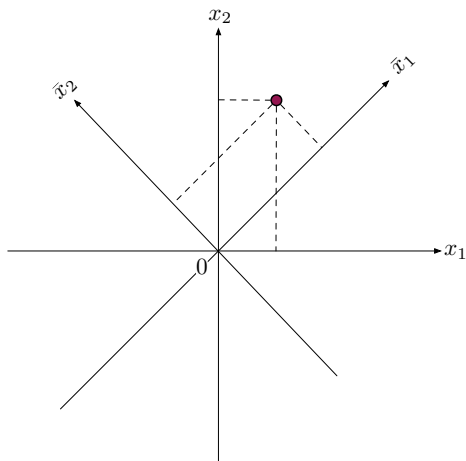
This is equivalent to having no RHP open-loop poles and no pole-zero cancellations in RHP.

Coordinate Transformations

Now that we have seen that a given transfer function can have many different state-space realizations, we would like a systematic procedure of generating such realizations, preferably with favorable properties (like controllability).

One such procedure is by means of *coordinate transformations*.

Coordinate Transformations



$$x \mapsto \bar{x} = Tx,$$

$$x = T^{-1}\bar{x}$$

$$T \in \mathbb{R}^{n \times n} \text{ nonsingular}$$

(go back and forth between the coordinate systems)

Coordinate Transformations

For example,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

This can be represented as

$$\bar{x} = Tx, \quad \text{where } T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The transformation is invertible: $\det T = -2$, and

$$T^{-1} = \frac{1}{\det T} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Or we can see this directly:

$$\bar{x}_1 + \bar{x}_2 = 2x_1; \quad \bar{x}_1 - \bar{x}_2 = 2x_2$$

Coordinate Transformations and State-Space Models

Consider a state-space model

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

and a change of coordinates $\bar{x} = Tx$ (T invertible).

What does the system look like in the new coordinates?

$$\dot{\bar{x}} = T\dot{x} = T\dot{x} \quad (\text{linearity of derivative})$$

$$= T(Ax + Bu)$$

$$= T(AT^{-1}\bar{x} + Bu) \quad (x = T^{-1}\bar{x})$$

$$= \underbrace{TAT^{-1}}_{\bar{A}}\bar{x} + \underbrace{TB}_{\bar{B}}u$$

$$y = Cx$$

$$= \underbrace{CT^{-1}}_{\bar{C}}\bar{x}$$

Coordinate Transformations and State-Space Models

$$\begin{array}{ccc} \dot{x} = Ax + Bu & \xrightarrow{T} & \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = Cx & & y = \bar{C}\bar{x} \end{array}$$

where

$$\bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}$$

What happens to

- ▶ the transfer function?
- ▶ the controllability matrix?

Coordinate Transformations and State-Space Models

$$\begin{array}{ccc} \dot{x} = Ax + Bu & \xrightarrow{T} & \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = Cx & & y = \bar{C}\bar{x} \end{array}$$

$$\text{where } \bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}$$

Claim: The transfer function doesn't change.

Proof:

$$\begin{aligned} \bar{G}(s) &= \bar{C}(Is - \bar{A})^{-1}\bar{B} \\ &= (CT^{-1})(Is - TAT^{-1})^{-1}(TB) \\ &= CT^{-1}(TIT^{-1}s - TAT^{-1})^{-1}TB \\ &= CT^{-1}[T(Is - A)T^{-1}]^{-1}TB \\ &= C \underbrace{T^{-1}T}_I (Is - A)^{-1} \underbrace{T^{-1}T}_I B \\ &= C(Is - A)^{-1}B \equiv G(s) \end{aligned}$$

Coordinate Transformations and State-Space Models

$$\begin{array}{ccc} \dot{x} = Ax + Bu & \xrightarrow{T} & \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = Cx & & y = \bar{C}\bar{x} \end{array}$$

$$\text{where } \bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}$$

The transfer function doesn't change.

In fact:

- ▶ open-loop poles don't change
- ▶ characteristic polynomial doesn't change:

$$\begin{aligned} \det(Is - \bar{A}) &= \det(Is - TAT^{-1}) \\ &= \det [T(Is - A)^{-1}T^{-1}] \\ &= \det T \cdot \det(Is - A)^{-1} \cdot \det T^{-1} \\ &= \det(Is - A)^{-1} \end{aligned}$$

Coordinate Transformations and State-Space Models

$$\begin{array}{ccc} \dot{x} = Ax + Bu & \xrightarrow{T} & \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = Cx & & y = \bar{C}\bar{x} \end{array}$$

$$\text{where } \bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}$$

Claim: Controllability doesn't change.

Proof: For any $k = 0, 1, \dots$,

$$\bar{A}^k \bar{B} = (TAT^{-1})^k TB = TA^k T^{-1} TB = TA^k B \quad (\text{by induction})$$

$$\begin{aligned} \text{Therefore, } \mathcal{C}(\bar{A}, \bar{B}) &= [TB \mid TAB \mid \dots \mid TA^{n-1}B] \\ &= T[B \mid AB \mid \dots \mid A^{n-1}B] \\ &= T\mathcal{C}(A, B) \end{aligned}$$

Since $\det T \neq 0$, $\det \mathcal{C}(\bar{A}, \bar{B}) \neq 0$ if and only if $\det \mathcal{C}(A, B) \neq 0$.

Thus, the new system is controllable if and only if the old one is.

Coordinate Transformations and State-Space Models

$$\begin{array}{ccc} \dot{x} = Ax + Bu & \xrightarrow{T} & \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = Cx & & y = \bar{C}\bar{x} \end{array}$$

$$\text{where } \bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}$$

Note: The *controllability matrix* does change:

$$\underbrace{\mathcal{C}(\bar{A}, \bar{B})}_{\text{new}} = \underbrace{T}_{\text{coord. trans.}} \underbrace{\mathcal{C}(A, B)}_{\text{old}}$$



$$T = \mathcal{C}(\bar{A}, \bar{B}) [\mathcal{C}(A, B)]^{-1}$$

This is a recipe for going from one *controllable* realization of a given t.f. to another.

CCF is the most convenient controllable realization of a given t.f., so we want to *convert a given controllable system to CCF* (useful for control design).

Example: Converting a Controllable System to CCF

Note!! The way I do this is different from the textbook.

Consider $A = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (C is immaterial).

Convert to CCF if possible.

Step 1: check for controllability.

$$AB = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -8 \end{pmatrix} \implies C = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}$$

$\det C = -1$ - controllable

Example: Converting a Controllable System to CCF

Step 2: Determine desired $\mathcal{C}(\bar{A}, \bar{B})$.

We need to figure out \bar{A} and \bar{B} .

For CCF, we must have

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so we need to find the coefficients a_1, a_2 .

Recall: the characteristic polynomial does not change:

$$\begin{aligned} \det(Is - A) &= \det(Is - \bar{A}) \\ \det \begin{pmatrix} s + 15 & -8 \\ 15 & s - 7 \end{pmatrix} &= \det \begin{pmatrix} s & -1 \\ a_2 & s + a_1 \end{pmatrix} \\ (s + 15)(s - 7) + 120 &= s(s + a_1) + a_2 \\ s^2 + 8s + 15 &= s^2 + a_1s + a_2 \end{aligned}$$

Example: Converting a Controllable System to CCF

Step 2: Determine desired $\mathcal{C}(\bar{A}, \bar{B})$.

We need to figure out \bar{A} and \bar{B} .

For CCF, we must have

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have just computed

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore, the new controllability matrix should be

$$\mathcal{C}(\bar{A}, \bar{B}) = [\bar{B} \mid \bar{A}\bar{B}] = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}$$

Example: Converting a Controllable System to CCF

Step 3: Compute T .

Recall: $T = \mathcal{C}(\bar{A}, \bar{B}) \cdot [\mathcal{C}(A, B)]^{-1}$

$$\mathcal{C}(A, B) = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}$$

$$\begin{aligned} [\mathcal{C}(A, B)]^{-1} &= \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}^{-1} \\ &= \frac{1}{-1} \begin{pmatrix} -8 & 7 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

$$\mathcal{C}(\bar{A}, \bar{B}) = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}$$

$$\begin{aligned} T &= \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

In the next lecture, we will see why CCF is so useful.