Plan of the Lecture

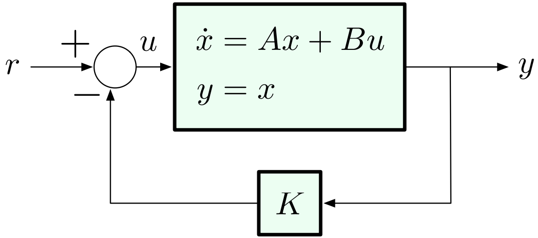
- Review: arbitrary pole placement by full state feedback.
- Today's topic: observer design for state estimation when full state feedback is not implementable.

Goal: for observable systems (definition to be introduced today), learn how to estimate the state x from output $y = Cx$ using an observer.

Reading: FPE, Chapter 7

Review: Pole Placement via State Feedback

Assume that the plant is controllable:



$$\dot{x} = Ax + B(-Kx + r) = (A - BK)x + Br, \qquad y = x$$

Transfer function from R to Y:

$$Y(s) = (Is - A + BK)^{-1} BR(s)$$

Closed-loop poles are the eigenvalues of $A - BK$!!

Review: Pole Placement in CCF

$$\dot{x} = (A - BK)x + Br, \qquad y = Cx$$

$$A - BK = -\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ a_n + k_1 & a_{n-1} + k_2 & \dots & a_2 + k_{n-1} & a_1 + k_n \end{pmatrix}$$

Closed-loop poles are the roots of the characteristic polynomial

$$\begin{aligned} &\det(Is - A + BK) \\ &= s^n + (a_1 + k_n)s^{n-1} + \ldots + (a_{n-1} + k_2)s + (a_n + k_1) \end{aligned}$$

Key observation: When the system is in CCF, each control gain affects only *one* of the coefficients of the characteristic polynomial, and these coefficients can be assigned arbitrarily by a suitable choice of $k_1, \ldots, k_n$.

Hence the name Controller Canonical Form — convenient for control design.

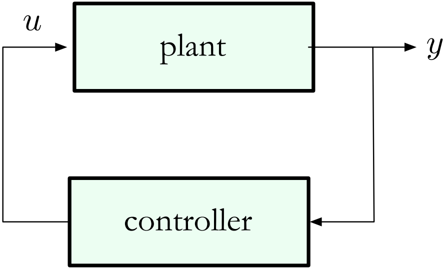
Pole Placement by State Feedback

General procedure for any *controllable* system:

1. Convert to CCF using a suitable invertible coordinate transformation T (such a transformation exists by controllability).
2. Solve the pole placement problem in the new coordinates.
3. Convert back to original coordinates.

Is Full State Feedback Always Available?

In a typical system, measurements are provided by sensors:



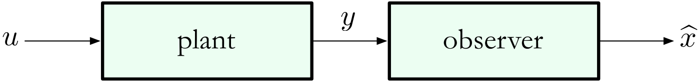
Full state feedback $u = -Kx$ is *not implementable*!!

When Full State Feedback Is Unavailable ...

... we need an Observer!!

State Estimation Using an Observer

When full state feedback is unavailable, the observer is used to estimate the state x:

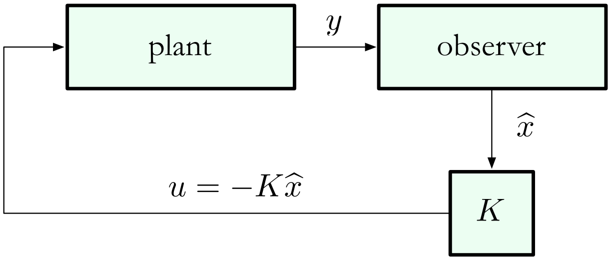


State Estimation Using an Observer

The idea is to design the observer in such a way that the state estimate $\widehat{x}$ is *asymptotically accurate*:

$$\|\widehat{x}(t) - x(t)\| = \sqrt{\sum_{i=1}^{n} \left(\widehat{x}_i(t) - x_i(t)\right)^2} \xrightarrow{t \to \infty} 0$$

If we are successful, then we can try estimated state feedback:



A New Concept: Observability

- Before, we saw that closed-loop poles can be assigned arbitrarily by full state feedback when the plant is controllable.
- Now, we will see that asymptotically accurate state estimation will be possible when the system is observable.
- Observability is a system property which is dual to controllability.

Observability

Consider a single-output system ($y \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Observability Matrix is defined as

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

— recall that C is $1 \times n$ and A is $n \times n$, so $\mathcal{O}(A, C)$ is $n \times n$;
— the observability matrix only involves A and C, not B

We say that the above system is observable if its observability matrix $\mathcal{O}(A, C)$ is *invertible*.

(This definition is only true for the single-output case; the multiple-output case involves the *rank* of $\mathcal{O}(A, C)$.)

Example: Computing $\mathcal{O}(A, C)$

$$\text{Let} \quad A = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Here, $n = 2$, $C \in \mathbb{R}^{1\times 2}$, $A \in \mathbb{R}^{2\times 2} \implies \mathcal{O}(A, C) \in \mathbb{R}^{2\times 2}$.

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \end{bmatrix}$$

where $CA = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & -5 \end{pmatrix}$

$$\therefore \ \mathcal{O}(A, C) = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}$$

$$\det \mathcal{O}(A, C) = -1 \qquad \implies \qquad \text{the system is observable}$$

— recall: this system is in Observer Canonical Form (OCF) ...

Observer Canonical Form

A single-output state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Observer Canonical Form (OCF) if the matrices A, C are of the form

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & * \\ 1 & 0 & \dots & 0 & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & * \\ 0 & 0 & \dots & 0 & 1 & * \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Fact: A system in OCF is *always observable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

Coordinate Transformations and Observability

Just like controllability, observability is preserved under invertible coordinate transformations.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \xrightarrow{T} \qquad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}, \qquad \bar{B} = TB, \qquad \bar{C} = CT^{-1}$

$$\begin{aligned} \mathcal{O}(\bar{A}, \bar{C}) &= \begin{pmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{pmatrix} = \begin{pmatrix} CT^{-1} \\ CT^{-1}TAT^{-1} \\ \vdots \\ CT^{-1}TA^{n-1}T^{-1} \end{pmatrix} \\ &= \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} T^{-1} = \mathcal{O}(A, C)T^{-1} \end{aligned}$$

Coordinate Transformations and Observability

Just like controllability, observability is preserved under invertible coordinate transformations:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \xrightarrow{T} \qquad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}$, $\qquad \bar{B} = TB$, $\qquad \bar{C} = CT^{-1}$

If the original system is observable, then

$$T \underbrace{\left[\mathcal{O}(A,C)\right]^{-1}}_{\text{old}} = \underbrace{\left[\mathcal{O}(\bar{A},\bar{C})\right]^{-1}}_{\text{new}}$$

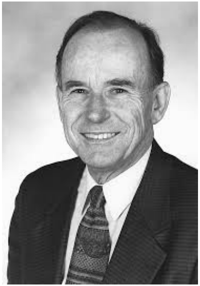
$$\Updownarrow$$

$$T = \underbrace{\left[\mathcal{O}(\bar{A},\bar{C})\right]^{-1}}_{\text{new}} \underbrace{\left[\mathcal{O}(A,C)\right]}_{\text{old}}$$

Observability and State Estimation

As we will show next:

If the system is observable, then there exists an observer (state estimator) that provides an asymptotically convergent estimate $\widehat{x}$ of the state x based on the observed output y.



The particular type of observer we will construct is called the Luenberger observer after David G. Luenberger, who developed this idea in his 1963 Ph.D. dissertation.

David Luenberger is a Professor at Stanford University.

The Luenberger Observer

Consider a state-space model

$$\begin{aligned} \dot{x} &= Ax \qquad\qquad \text{(for now, assume } u = 0\text{)} \\ y &= Cx \end{aligned}$$

We wish to estimate the state x based on the output y.

Consider feeding the output y as input to the following system with state $\widehat{x}$:

$$\dot{\widehat{x}} = (A - LC)\widehat{x} + Ly.$$

Assumption: The output injection matrix L is chosen in such a way that the matrix $A - LC$ is Hurwitz (i.e., all of its eigenvalues lie in LHP).

At this point, we do not assume anything about observability.

The Luenberger Observer

$$\begin{aligned}
\text{System:} \qquad & \dot{x} = Ax \\
& y = Cx \\
\text{Observer:} \qquad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly.
\end{aligned}$$

What happens to state estimation error $e = x - \widehat{x}$ as $t \to \infty$?

$$\begin{aligned}
\dot{e} &= \dot{x} - \dot{\widehat{x}} \\
&= Ax - [(A - LC)\widehat{x} + LCx] \\
&= (A - LC)x - (A - LC)\widehat{x} \\
&= (A - LC)e
\end{aligned}$$

Does $e(t)$ converge to zero in some sense?

Linear ODEs and Eigenvalues: A Digression

$$\dot{v} = Fv, \qquad v \in \mathbb{R}^n,\ F \in \mathbb{R}^{n \times n}$$

Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of F, i.e., roots of $\det(Is - F) = 0$.

Then there exists a matrix $T \in \mathbb{R}^{n \times n}$, such that $T^{-1} = T^T$ and

$$F = T^{-1} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} T$$

Consider the change of coordinates $\bar{v} = Tv$. Then

$$\dot{\bar{v}} = TFT^{-1}\bar{v} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \bar{v}$$

Linear ODEs: A Digression

$$\dot{\bar{v}} = TFT^{-1}\bar{v} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \bar{v}, \qquad (\lambda_1, \ldots, \lambda_n) = \mathrm{eig}(F)$$

$$\Updownarrow$$

$$\dot{\bar{v}}_i = \lambda_i \bar{v}_i, \qquad i = 1, 2, \ldots, n$$

This system of n 1st-order ODEs has the solution

$$\bar{v}_i(t) = \bar{v}_i(0) e^{\lambda_i t}, \qquad i = 1, 2, \ldots, n$$

If all λ_i's have negative real parts, then

$$\begin{aligned} \|v(t)\|^2 &= v(t)^T v(t) = \bar{v}(t)^T \bar{v}(t) \\ &\leq Ce^{-2\sigma_{\min} t}, \qquad \text{where } \sigma_{\min} = \min_{1 \leq i \leq n} |\mathrm{Re}(\lambda_i)| \end{aligned}$$

The Luenberger Observer

$$
\begin{aligned}
\text{System:} \qquad & \dot{x} = Ax \\
& y = Cx \\
\text{Observer:} \qquad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly \\
\text{Error:} \qquad & \dot{e} = (A - LC)e
\end{aligned}
$$

Recall our assumption that $A - LC$ is Hurwitz (all eigenvalues are in LHP). This implies that

$$\|x(t) - \widehat{x}(t)\|^2 = \|e(t)\|^2 = \sum_{i=1}^{n} |e_i(t)|^2 \xrightarrow{t \to \infty} 0$$

at an exponential rate, determined by the eigenvalues of $A - LC$.

For fast convergence, want eigenvalues of $A - LC$ far into LHP!!

The Luenberger Observer

$$\begin{aligned}
\text{System:} \qquad & \dot{x} = Ax \\
& y = Cx \\
\text{Observer:} \qquad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly \\
\text{Error:} \qquad & \dot{e} = (A - LC)e
\end{aligned}$$

Observer transfer function:

$$\begin{aligned}
& s\widehat{X}(s) = (A - LC)\widehat{X}(s) + LY(s) \\
& (Is - A + LC)\widehat{X}(s) = LY(s) \\
& \widehat{X}(s) = (Is - A + LC)^{-1}LY(s).
\end{aligned}$$

The eigenvalues of $A - LC$ are the observer poles. We want these poles to be *stable* and *fast*.

Observability and Estimation Error

Fact: If the system

$$\dot{x} = Ax, \qquad y = Cx$$

is observable, then we can arbitrarily assign eigenvalues of $A - LC$ by a suitable choice of the output injection matrix L.

This is similar to the fact that controllability implies arbitrary closed-loop pole placement by state feedback.

In fact, these two facts are closely related because CCF is dual to OCF.

Observer Pole Placement in OCF

Consider a single-output system in OCF:

$$\dot{x} = Ax$$
$$y = Cx, \qquad y \in \mathbb{R}$$

where $A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -a_n \\ 1 & 0 & \dots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_2 \\ 0 & 0 & \dots & 0 & 1 & -a_1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}$

Note that A^T has the form of a CCF system matrix, thus:

$$\begin{aligned} \det(Is - A) &= \det((Is - A)^T) = \det(Is - A^T) \\ &= s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n \end{aligned}$$

Now Let's Add an Observer

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_n \\ 0 & 1 & \dots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & \dots & 1 & -a_1 \end{pmatrix}$$

$$LC = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & \ell_1 \\ 0 & 0 & \dots & 0 & \ell_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \ell_{n-1} \\ 0 & 0 & \dots & 0 & \ell_n \end{pmatrix}$$

$$A - LC = \begin{pmatrix} 0 & 0 & \dots & 0 & -(a_n + \ell_1) \\ 0 & 1 & \dots & 0 & -(a_{n-1} + \ell_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -(a_2 + \ell_{n-1}) \\ 0 & 0 & \dots & 1 & -(a_1 + \ell_n) \end{pmatrix}$$

— still in OCF!!

Observer Pole Placement in OCF

$$\dot{x} = Ax, \qquad y = Cx, \qquad \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly$$

$$A - LC = \begin{pmatrix} 0 & 0 & \dots & 0 & -(a_n + \ell_1) \\ 1 & 0 & \dots & 0 & -(a_{n-1} + \ell_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -(a_2 + \ell_{n-1}) \\ 0 & 0 & \dots & 1 & -(a_1 + \ell_n) \end{pmatrix}$$

Eigenvalues of $A - LC$ are the roots of the characteristic polynomial

$$\begin{aligned} &\det(Is - A + LC) \\ &= s^n + (a_1 + \ell_n)s^{n-1} + \ldots + (a_{n-1} + \ell_2)s + (a_n + \ell_1) \end{aligned}$$

Key observation: In OCF, each observer gain affects only *one* of the coefficients of the characteristic polynomial, which can be assigned arbitrarily by a suitable choice of $\ell_1, \ldots, \ell_n$.

Hence the name Observer Canonical Form — convenient for observer design.

Observer Pole Placement

General procedure for any *observable* system:

1. Convert to OCF: $T = \underbrace{\mathcal{O}(\bar{A}, \bar{C})^{-1}}_{\text{new}} \underbrace{[\mathcal{O}(A, C)]}_{\text{old}}$
2. Find $\bar{L}$, such that $\bar{A} - \bar{L}\bar{C}$ has desired eigenvalues.
3. Convert back to original coordinates: $L = T^{-1}\bar{L}$.

The resulting observer is

$$\dot{\widehat{x}} = (A - T^{-1}\bar{L}C)\widehat{x} + T^{-1}\bar{L}y$$

In fact, this procedure is not necessary because of duality between controllability and observability!!

Controllability–Observability Duality

Claim: The system

$$\dot{x} = Ax, \qquad y = Cx$$

is observable if and only if the system

$$\dot{x} = A^T x + C^T u$$

is controllable.

Proof:
$$\begin{aligned}\mathcal{C}(A^T, C^T) &= \left[C^T \mid A^T C^T \mid \ldots \mid (A^T)^{n-1} C^T\right] \\ &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^T = \left[\mathcal{O}(A, C)\right]^T\end{aligned}$$

Thus, $\mathcal{O}(A, C)$ is nonsingular if and only if $\mathcal{C}(A^T, C^T)$ is.

Observer Pole Placement, O/C Duality Version

Given an observable pair (A, C):

1. For $F = A^T$, $G = C^T$, consider the system $\dot{x} = Fx + Gu$ (this system is controllable).
2. Use our earlier procedure to find K, such that

$$F - GK = A^T - C^T K$$

   has desired eigenvalues.
3. Then

$$\text{eig}(A^T - C^T K) = \text{eig}(A^T - C^T K)^T = \text{eig}(A - K^T C),$$

   so $L = K^T$ is the desired output injection matrix.

Final answer: use the observer

$$\begin{aligned}\dot{\hat{x}} &= (A - LC)\hat{x} + Ly \\ &= (A - K^T C)\hat{x} + K^T y.\end{aligned}$$

Combining Full-State Feedback with an Observer

- So far, we have focused on autonomous systems ($u = 0$).
- What about nonzero inputs?

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

— assume (A, B) is controllable and (A, C) is observable.

- In the next lecture, we will learn how to use an observer together with estimated state feedback to (approximately) place closed-loop poles.