Plan of the Lecture

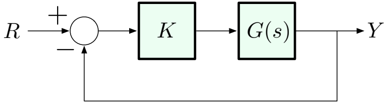
- Review: Bode plots for three types of transfer functions
- Today's topic: stability from frequency response; gain and phase margins

Goal: learn to read off stability properties of the closed-loop system from the Bode plot of the open-loop transfer function; define and calculate Gain and Phase Margins, important quantitative measures of "distance to instability."

Reading: FPE, Section 6.1

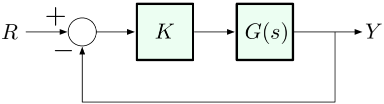
Stability from Frequency Response

Consider this unity feedback configuration:



Question: How can we decide whether the *closed-loop* system is stable for a given value of $K > 0$ based on our knowledge of the *open-loop* transfer function $KG(s)$?

Stability from Frequency Response



Question: How can we decide whether the *closed-loop* system is stable for a given value of $K > 0$ based on our knowledge of the *open-loop* transfer function $KG(s)$?

One answer: use root locus.

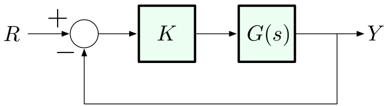
Points on the root locus satisfy the characteristic equation

$$1 + KG(s) = 0 \qquad \Longleftrightarrow \qquad KG(s) = -1 \qquad \left(\Longleftrightarrow G(s) = -\frac{1}{K} \right)$$

If $s \in \mathbb{C}$ is on the RL, then

$$|KG(s)| = 1 \qquad \text{and} \qquad \angle KG(s) = \angle G(s) = 180^\circ \mod 360^\circ$$

Stability from Frequency Response



Question: How can we decide whether the *closed-loop* system is stable for a given value of $K > 0$ based on our knowledge of the *open-loop* transfer function $KG(s)$?

Another answer: let's look at the Bode plots:

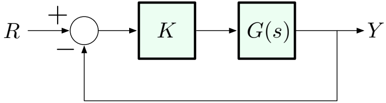
$$\omega \longmapsto |KG(j\omega)| \qquad \text{on log-log scale}$$

$$\omega \longmapsto \angle KG(j\omega) \qquad \text{on log-linear scale}$$

— Bode plots show us magnitude and phase, but only for $s = j\omega$, $0 < \omega < \infty$

How does this relate to the root locus? $j\omega$-crossings!!

Stability from Frequency Response



Stability from frequency response. If $s = j\omega$ is on the root locus (for some value of K), then

$$|KG(j\omega)| = 1 \qquad \text{and} \qquad \angle KG(j\omega) = 180^\circ \mod 360^\circ$$

Therefore, the transition from stability to instability can be detected in two different ways:

- from root locus — as $j\omega$-crossings
- from Bode plots — as $M = 1$ and $\phi = 180^\circ$ at some frequency ω (for a given value of K)

Example

$$KG(s) = \frac{K}{s(s^2 + 2s + 2)}$$

Characteristic equation:

$$1 + \frac{K}{s(s^2 + 2s + 2)} = 0$$
$$s(s^2 + 2s + 2) + K = 0$$
$$s^3 + 2s^2 + 2s + K = 0$$

Recall the necessary & sufficient condition for stability for a 3rd-degree polynomial $s^3 + a_1 s^2 + a_2 s + a_3$:

$$a_1, a_2, a_3 > 0, \qquad a_1 a_2 > a_3.$$

Here, the closed-loop system is stable if and only if $0 < K < 4$.

Let's see what we can read off from the Bode plots.

Example, continued

$$KG(s) = \frac{K}{s(s^2 + 2s + 2)}$$

$$\text{Bode form:} \quad KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

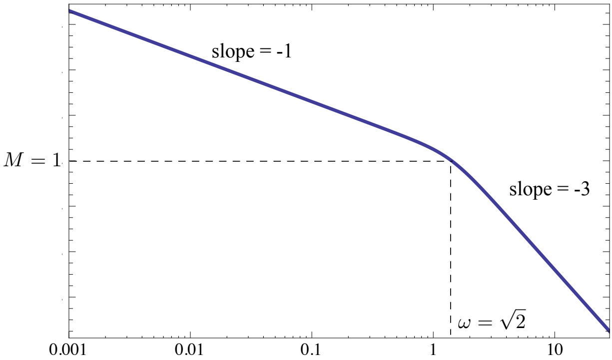
Plot the magnitude first:

- Type 1 (low-frequency) asymptote: $\dfrac{K/2}{j\omega}$
  $K_0 = K/2, \;\; n = -1 \;\; \Longrightarrow$ slope $= -1$, passes through $(\omega = 1, M = K/2)$
- Type 3 (complex pole) asymptote:
  break-point at $\omega = \sqrt{2} \;\; \Longrightarrow$ slope down by 2
- $\zeta = \dfrac{1}{\sqrt{2}} \Longrightarrow$ no reasonant peak

Example, Magnitude Plot

$$KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Magnitude plot for $K = 4$ (the critical value):

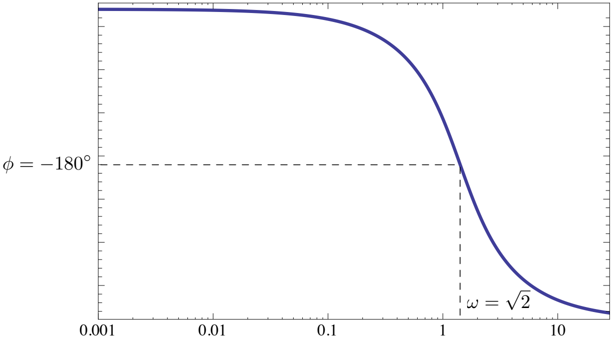


When $\omega = \sqrt{2}$, $M = |4G(j\omega)| = \left|\frac{2}{j\sqrt{2}\left(j^2 + j\sqrt{2} + 1\right)}\right| = 1$

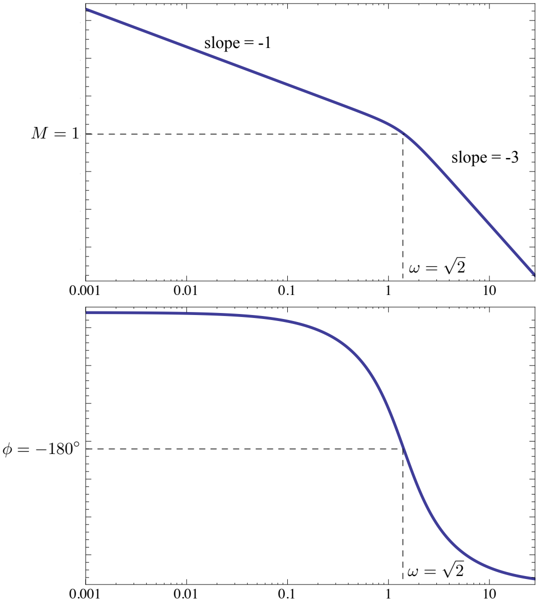
Example, Phase Plot

$$KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Phase plot (independent of K):



When $\omega = \sqrt{2}$, $\phi = -180°$

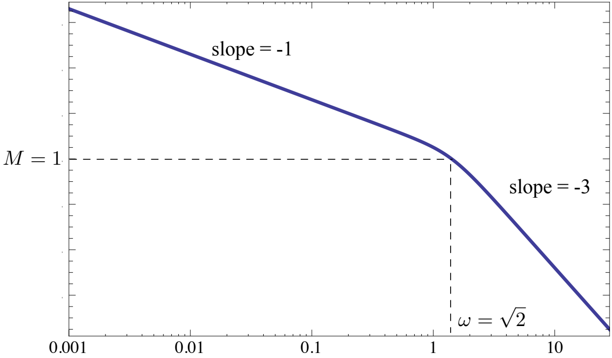


For the critical value $K = 4$:

$M = 1$ and $\phi = 180°$ mod $360°$ at $\omega = \sqrt{2}$

Crossover Frequency and Stability

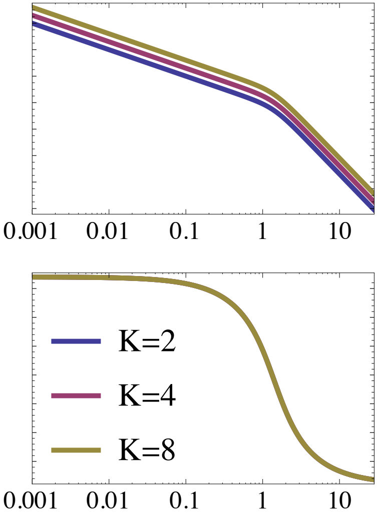
Definition: The frequency at which $M = 1$ is called the *crossover frequency* and denoted by ω_c.



Transition from stability to instability on the Bode plot:

$$\text{for critical } K, \qquad \angle G(j\omega_c) = 180^\circ$$

Effect of Varying K



What happens as we vary K?

- ϕ independent of $K \implies$ only the M-plot changes
- If we multiply K by 2:

$$\log(2M) = \log 2 + \log M$$

  – M-plot shifts up by $\log 2$

- If we divide K by 2:

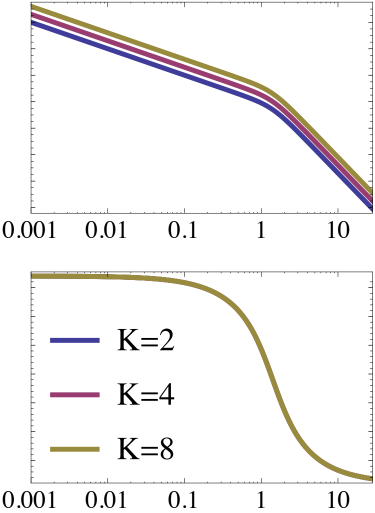
$$\log(\frac{1}{2}M) = \log\frac{1}{2} + \log M$$
$$= -\log 2 + \log M$$

  – M-plot shifts down by $\log 2$

Changing the value of K moves the crossover frequency ω_c!!

Effect of Varying K

Changing the value of K moves the crossover frequency ω_c!!

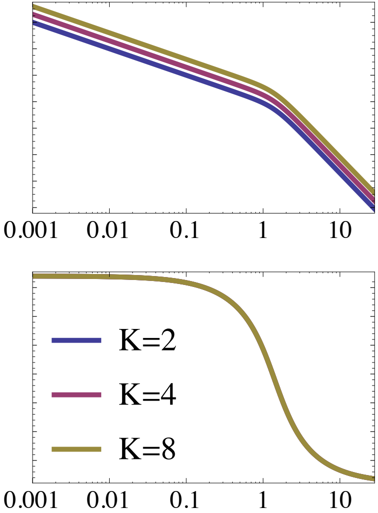


What happens as we vary K?

$$\angle KG(j\omega_c) \begin{cases} > -180^\circ, & \text{for } K < 4 \\ & \text{(stable)} \\ = -180^\circ, & \text{for } K = 4 \\ & \text{(critical)} \\ < -180^\circ, & \text{for } K > 4 \\ & \text{(unstable)} \end{cases}$$

Effect of Varying K

Changing the value of K moves the crossover frequency ω_c!!



Equivalently, we may define ω_{180° as the frequency at which

$$\phi = 180^\circ \mod 360^\circ.$$

Then, *in this example**,

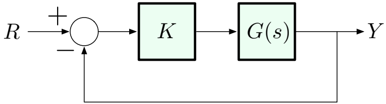
$$|KG(j\omega_{180^\circ})| < 1 \quad \longleftrightarrow \quad \text{stability}$$
$$|KG(j\omega_{180^\circ})| > 1 \quad \longleftrightarrow \quad \text{instability}$$

* Not a general rule; conditions will

vary depending on the system, must use either root locus or Nyquist plot to resolve ambiguity.

Stability from Frequency Response

Consider this unity feedback configuration:



Suppose that the *closed-loop* system, with transfer function

$$\frac{KG(s)}{1 + KG(s)},$$

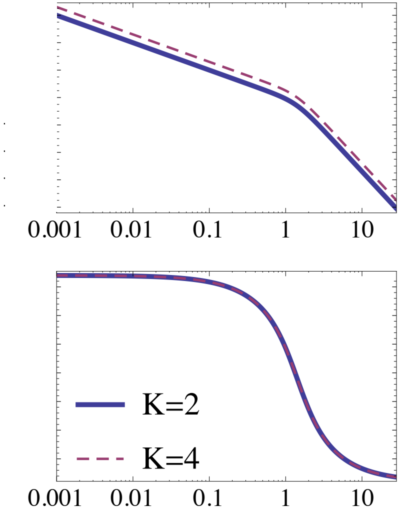
is stable for a given value of K.

Question: Can we use the Bode plot to determine how far from instability we are?

Two important characteristics: gain margin (GM) and phase margin (PM).

Gain Margin

Back to our example: $G(s) = \frac{1}{s(s^2+2s+2)}$, $K = 2$ (stable)

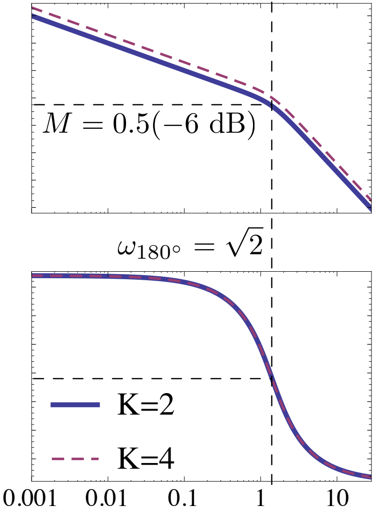


Gain margin (GM) is the factor by which K can be multiplied before we get $M = 1$ when $\phi = 180°$

Since varying K doesn't change $\omega_{180°}$, to find GM we need to inspect M at $\omega = \omega_{180°}$

Gain Margin

Our example: $G(s) = \frac{1}{s(s^2+2s+2)}$, $K = 2$ (stable)



Gain margin (GM) is the factor by which K can be multiplied before we get $M = 1$ when $\phi = 180^\circ$

Since varying K doesn't change ω_{180°, to find GM we need to inspect M at $\omega = \omega_{180^\circ}$

In this example:

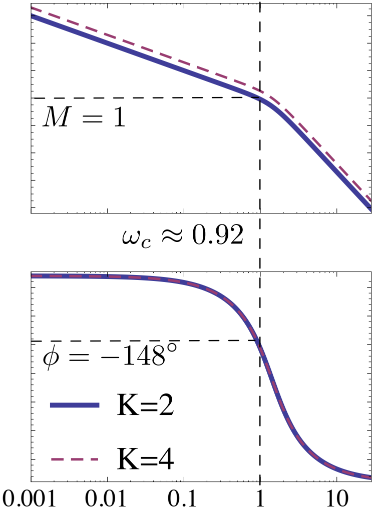
at $\omega_{180^\circ} = \sqrt{2}$

$M = 0.5\,(-6\text{ dB})$,

so $\text{GM} = 2$

Phase Margin

Our example: $G(s) = \dfrac{1}{s(s^2+2s+2)}$, $K = 2$ (stable)



Phase margin (PM) is the amount by which the phase at the crossover frequency ω_c differs from $180^\circ \mod 360^\circ$

To find PM, we need to inspect ϕ at $\omega = \omega_c$

In this example:

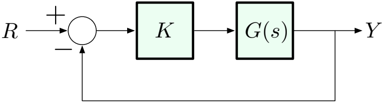
at $\omega_c \approx 0.92$

$$\phi = -148^\circ,$$

so $\text{PM} = (-148^\circ) - (-180^\circ) = 32^\circ$

(in practice, want $\text{PM} \geq 30^\circ$)

Example 2



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} \qquad\qquad \zeta, \omega_n > 0$$

Consider gain $K = 1$, which gives closed-loop transfer function

$$\frac{KG(s)}{1 + KG(s)} = \frac{\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s}}{1 + \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s}}$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$ — prototype 2nd-order response

Question: what is the gain margin at $K = 1$?

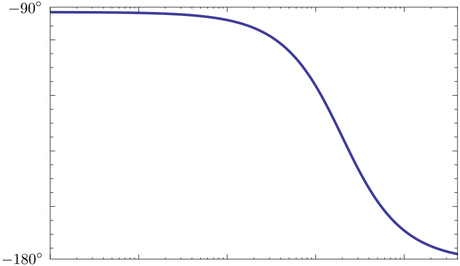
Answer: $\mathrm{GM} = \infty$

Example 2

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

Let's look at the phase plot:

- starts at -90° (Type 1 term with $n = -1$)
- goes down by -90° (Type 2 pole)



Recall: to find GM, we first need to find ω_{180°, and here there is no such $\omega \implies$ no GM.

Example 2

So, at $K = 1$, the gain margin of

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

is equal to ∞ — what does that mean?

It means that we can keep on increasing K indefinitely without ever encountering instability.

But we already knew that: the characteristic polynomial is

$$p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2,$$

which is *always stable*.

What about phase margin?

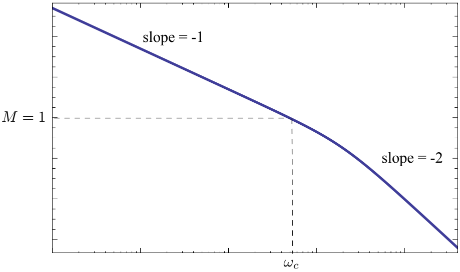
Example 2: Phase Margin

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

Let's look at the magnitude plot:

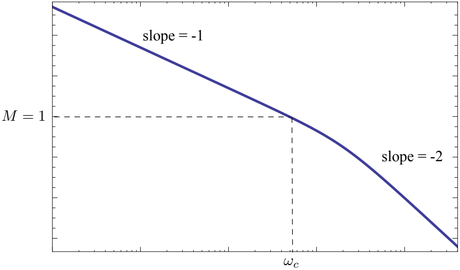
- low-frequency asymptote slope -1 (Type 1 term, $n = -1$)
- slope down by 1 past the breakpt. $\omega = 2\zeta\omega_n$ (Type 2 pole)

$\Longrightarrow$ there is a finite crossover frequency ω_c!!



Example 2: Magnitude Plot

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$



It can be shown that, *for this system*,

$$\mathrm{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{4\zeta^4+1} - 2\zeta^2}\right)$$

— for PM $< 70^\circ$, a good approximation is PM $\approx 100 \cdot \zeta$

Phase Margin for 2nd-Order System

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

$$\text{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{4\zeta^4+1} - 2\zeta^2}\right) \approx 100 \cdot \zeta$$

Conclusions:

larger PM $\Longleftrightarrow$ better damping

(open-loop quantity) (closed-loop characteristic)

Thus, the overshoot $M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ and resonant peak $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$ are both related to PM through ζ!!

Preview: Bode's Gain-Phase Relationship

In the next lecture, we will see the following more generally:

Hendrik Wade Bode
(1905–1982)

Bode's Gain-Phase Relationship: all important characteristics of the closed-loop time response can be related to the phase margin of the open-loop transfer function!!

In fact, we will use a quantitative statement of this relationship as a design guideline.