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- Review: observability; Luenberger observer and state estimation error.
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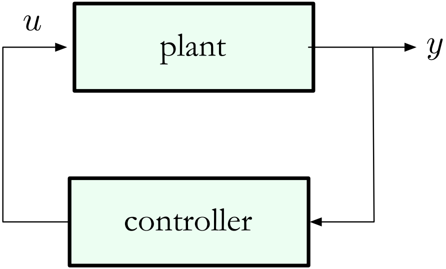
Reading: FPE, Chapter 7

Is Full State Feedback Always Available?

In a typical system, measurements are provided by sensors:



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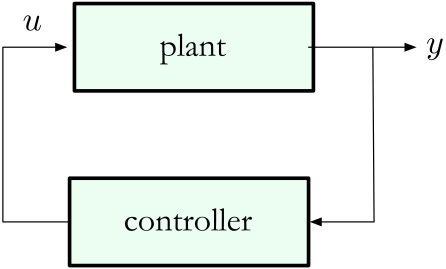
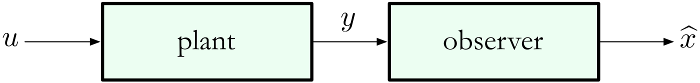


Full state feedback $u = -Kx$ is *not implementable*!!

In that case, an observer is used to estimate the state x:



State Estimation Using an Observer

If the system is observable, the state estimate $\widehat{x}$ is *asymptotically accurate*:

$$\|\widehat{x}(t) - x(t)\| = \sqrt{\sum_{i=1}^{n} \left(\widehat{x}_i(t) - x_i(t)\right)^2} \xrightarrow{t\to\infty} 0$$

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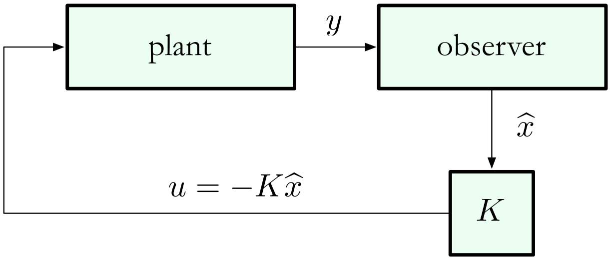
If we are successful, then we can try estimated state feedback:



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Consider a single-output system ($y \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

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(This definition is only true for the single-output case; the multiple-output case involves the *rank* of $\mathcal{O}(A, C)$.)

Observer Canonical Form

A single-output state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Observer Canonical Form (OCF) if the matrices A, C are of the form

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & * \\ 1 & 0 & \dots & 0 & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & * \\ 0 & 0 & \dots & 0 & 1 & * \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

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Fact: A system in OCF is *always observable*!!

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(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

The Luenberger Observer

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\begin{aligned}
\text{System:} \qquad & \dot{x} = Ax \\
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Does $e(t)$ converge to zero in some sense?

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Recall our assumption that $A - LC$ is Hurwitz (all eigenvalues are in LHP). This implies that

$$\|x(t) - \widehat{x}(t)\|^2 = \|e(t)\|^2 = \sum_{i=1}^{n} |e_i(t)|^2 \xrightarrow{t \to \infty} 0$$

at an exponential rate, determined by the eigenvalues of $A - LC$.

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For fast convergence, want eigenvalues of $A - LC$ far into LHP!!

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In fact, these two facts are closely related because CCF is dual to OCF.

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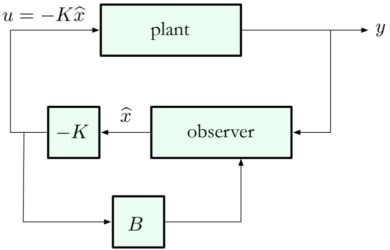

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- We know how to find K, such that $A - BK$ has desired eigenvalues (controller poles).
- Since we do not have access to x, we must design an observer. But this time, we need a slight modification because of the Bu term.

Observer in the Presence of Control Input

- Let's see what goes wrong when we use the old approach:

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Observer and Controller

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- By controllability, we can arbitrarily assign $\text{eig}(A - BK)$.

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The overall observer-controller system is:

$$\begin{aligned} \dot{\widehat{x}} &= (A - LC)\widehat{x} + Ly + B\underbrace{(-K\widehat{x})}_{=u} \\ &= (A - LC - BK)\widehat{x} + Ly \\ u &= -K\widehat{x} \end{aligned}$$

(dynamic output feedback)

— this is a dynamical system with input y and output u

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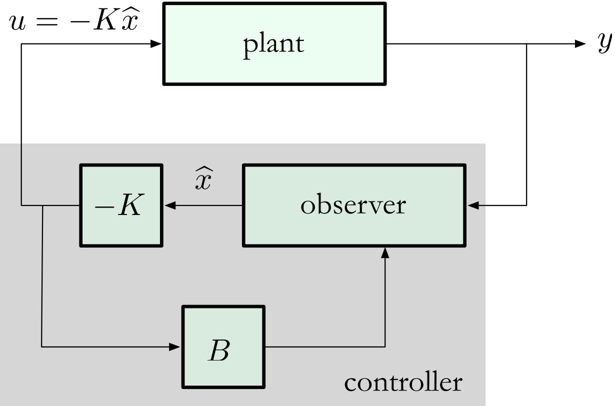

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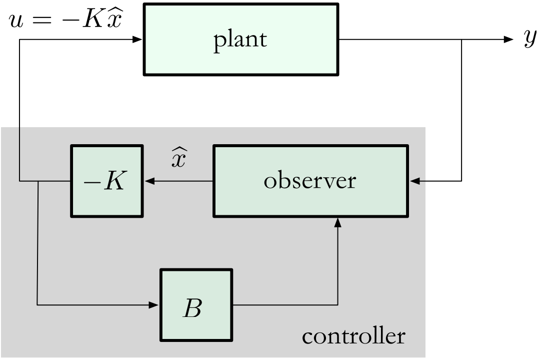

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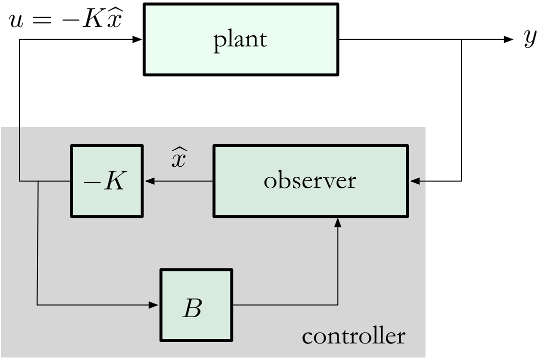

Controller transfer function (from y to u):

$$s\widehat{X} = (A - LC - BK)\widehat{X} + LY, \quad U = -K\widehat{X}$$

$$U = \underbrace{-K(Is - A + LC + BK)^{-1}L}_{=D(s)}Y$$

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We can write it in block matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{\widehat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A - LC - BK \end{pmatrix} \begin{pmatrix} x \\ \widehat{x} \end{pmatrix}$$

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How do we relate this to "nominal" behavior, $A - BK$?

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Let us transform to new coordinates:

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The Main Result: Separation Principle

So now we can write

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \underbrace{\begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix}}_{\text{upper triangular matrix}} \begin{pmatrix} x \\ e \end{pmatrix}$$

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— this holds only for linear systems!!

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- If we choose observer poles to be several times faster than the controller poles (e.g., 2–5 times), then the controller poles will be dominant.

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- Remember: the system must be controllable and observable!!