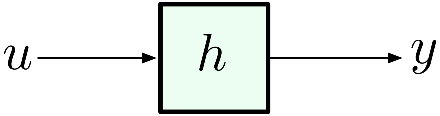
Plan of the Lecture

- Review: dynamic response; transfer functions; transient and steady-state response
- Today's topic: dynamic response (transient and steady-state) with arbitrary I.C.'s

Goal: develop a methodology for characterizing the output of a given system for a given input.

Reading: FPE, Section 3.1, Appendix A

Dynamic Response



Problem: compute the response y to a given input u under a given set of initial conditions.

In particular, we wish to know both the transient response (due to I.C.'s) and the steady-state response (once the effect of the I.C.'s "washes away").

Laplace Transforms Revisited

(see FPE, Appendix A)

One-sided (or unilateral) Laplace transform:

$$\mathscr{L}\{f(t)\} \equiv F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \qquad \text{(really, from } 0^-\text{)}$$

— for simple functions f, can compute $\mathscr{L}f$ by hand.

Example: unit step

$$f(t) = 1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathscr{L}\{1(t)\} = \int_0^\infty e^{-st}\mathrm{d}t = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \qquad \text{(pole at } s = 0\text{)}$$

— this is valid provided $\mathrm{Re}(s) > 0$, so that $e^{-st} \xrightarrow{t\to+\infty} 0$.

Laplace Transforms Revisited

Example: $f(t) = \cos t$

$$\mathscr{L}\{\cos t\} = \mathscr{L}\left\{\frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}\right\} \qquad \text{(Euler's formula)}$$

$$= \frac{1}{2}\mathscr{L}\{e^{jt}\} + \frac{1}{2}\mathscr{L}\{e^{-jt}\} \qquad \text{(linearity)}$$

$$\mathscr{L}\{e^{jt}\} = \int_0^\infty e^{jt}e^{-st}\mathrm{d}t = \int_0^\infty e^{(j-s)t}\mathrm{d}t = \frac{1}{j-s}e^{(j-s)t}\Big|_0^\infty$$

$$= -\frac{1}{j-s} \qquad \text{(pole at } s = j\text{)}$$

$$\mathscr{L}\{e^{-jt}\} = \int_0^\infty e^{-jt}e^{-st}\mathrm{d}t = \int_0^\infty e^{-(j+s)t}\mathrm{d}t = -\frac{1}{j+s}e^{-(j+s)t}\Big|_0^\infty$$

$$= \frac{1}{j+s} \qquad \text{(pole at } s = -j\text{)}$$

— in both cases, require $\mathrm{Re}(s) > 0$, i.e., s must lie in the right half-plane (RHP)

Laplace Transforms Revisited

Example: $f(t) = \cos t$

$$\begin{aligned}
\mathscr{L}\{\cos t\} &= \frac{1}{2}\mathscr{L}\{e^{jt}\} + \frac{1}{2}\mathscr{L}\{e^{-jt}\} \\
&= \frac{1}{2}\left(-\frac{1}{j-s} + \frac{1}{j+s}\right) \\
&= \frac{1}{2}\left(\frac{-\cancel{j} - s + \cancel{j} - s}{(j-s)(j+s)}\right) \\
&= \frac{1}{2}\left(\frac{-2s}{-1 + \cancel{js} - \cancel{js} - s^2}\right) \\
&= \frac{s}{s^2+1} \qquad \text{(poles at } s = \pm j\text{)}
\end{aligned}$$

for $\mathrm{Re}(s) > 0$

Laplace Transforms Revisited

Convolution: $\mathscr{L}\{f \star g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$
(useful because $Y(s) = H(s)U(s)$)

Example: $\dot{y} = -y + u \qquad y(0) = 0$

Compute the response for $u(t) = \cos t$

We already know

$$H(s) = \frac{1}{s+1} \qquad \text{(from earlier example)}$$
$$U(s) = \frac{s}{s^2+1} \qquad \text{(just proved)}$$

$$\Longrightarrow Y(s) = H(s)U(s) = \frac{s}{(s+1)(s^2+1)}$$
$$y(t) = \mathscr{L}^{-1}\{Y\}$$

— can't find $Y(s)$ in the tables. So how do we compute y?

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\left\{\frac{s}{(s+1)(s^2+1)}\right\}$

This Laplace transform is not in the tables, but let's look at the table anyway. What do we find?

$$\frac{1}{s+1} \qquad \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \qquad (\#7)$$

$$\frac{1}{s^2+1} \qquad \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t \qquad (\#17)$$

$$\frac{s}{s^2+1} \qquad \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t \qquad (\#18)$$

— so we see some things that are similar to $Y(s)$, but not quite.

This brings us to the method of partial fractions:

- boring (i.e., character-building), but *very useful*
- allows us to break up complicated fractions into sums of simpler ones, for which we know $\mathscr{L}^{-1}$ from tables

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We seek a, b, c, such that

$$Y(s) = \frac{a}{s+1} + \frac{bs+c}{s^2+1} \quad \text{(need } bs+c \text{ so that deg(num) = deg(den) } - 1)$$

- Find a: multiply by $s+1$ to isolate a

$$(s+1)Y(s) = \frac{s}{s^2+1} = a + \frac{(s+1)(as+b)}{(s^2+1)}$$

— now let $s = -1$ to "kill" the second term on the RHS:

$$a = (s+1)Y(s)\Big|_{s=-1} = -\frac{1}{2}$$

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We seek a, b, c, such that

$$Y(s) = \frac{a}{s+1} + \frac{bs+c}{s^2+1} \quad \text{(need } bs+c \text{ so that } \deg(\text{num}) = \deg(\text{den}) - 1\text{)}$$

- Find b: multiply by s^2+1 to isolate $bs+c$

$$(s^2+1)Y(s) = \frac{s}{s+1} = \frac{a(s^2+1)}{s+1} + bs + c$$

— now let $s = j$ to "kill" the first term on the RHS:

$$bj + c = (s^2+1)Y(s)\Big|_{s=j} = \frac{j}{1+j}$$

Match $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ parts:

$$c + bj = \frac{j}{1+j} = \frac{j(1-j)}{(1+j)(1-j)} = \frac{1}{2} + \frac{j}{2} \quad \Longrightarrow b = c = \tfrac{1}{2}$$

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We found that

$$Y(s) = -\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}$$

Now we can use linearity and tables:

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left\{-\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}\right\} \\
&= -\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{(from tables)} \\
&= -\frac{1}{2}e^{-t} + \frac{1}{\sqrt{2}}\cos(t - \pi/4) \quad (\cos(a-b) = \cos a \cos b + \sin a \sin b)
\end{aligned}$$

Transient and Steady-State Response

Consider the system $\dot{y} = -y + u \qquad y(0) = 0$

$$u(t) = \cos t \quad \longrightarrow \quad y(t) = \underbrace{-\frac{1}{2}e^{-t}}_{\substack{\text{transient}\\ \text{response}}} + \underbrace{\frac{1}{\sqrt{2}}\cos(t - \pi/4)}_{\substack{\text{steady-state}\\ \text{response}}}$$

— transient response vanishes as $t \to \infty$ (we will see later why)

Let's compare against the frequency response formula:

$$H(s) = \frac{1}{s+1} \qquad \Longrightarrow \qquad H(j\omega) = \frac{1}{j\omega + 1}$$

$u(t) = \cos t$ has $A = 1$ and $\omega = 1$, so

$$\begin{aligned} y(t) &= M(1)\cos\big(t + \varphi(1)\big) \\ &= \frac{1}{\sqrt{2}}\cos\big(t - \pi/4\big) \end{aligned}$$

— the freq. response formula gives only the steady-state part!!

Transient and Steady-State Response

Consider the system $\dot{y} = -y + u \qquad y(0) = 0$

We computed the response to $u(t) = \cos t$ in two ways:

$$y(t) = -\frac{1}{2}e^{-t} + \frac{1}{\sqrt{2}}\cos(t - \pi/4)$$

— using the method of partial fractions;

$$y(t) = \frac{1}{\sqrt{2}}\cos\left(t - \pi/4\right)$$

— using the frequency response formula.

Q: Which answer is correct? And why?

A: At $t = 0$, $\frac{1}{\sqrt{2}}\cos(t - \pi/4) = \frac{1}{2} \neq 0$, which is inconsistent with the initial condition $y(0) = 0$. The term $-\frac{1}{2}e^{-t}\Big|_{t=0} = -\frac{1}{2}$ cancels the steady-state term, so indeed $y(0) = 0$.

Therefore, the first formula is correct.

Transient and Steady-State Response

Main message: the frequency response formula only gives the steady-state part of the response, but the inverse Laplace transform gives the whole response (including the transient part).

— we will now see how to deal with nonzero I.C.'s ...

Laplace Transforms and Differentiation

Given a differentiable function f, what is the Laplace transform $\mathscr{L}\{f'(t)\}$ of its time derivative?

$$\begin{aligned}\mathscr{L}\{f'(t)\} &= \int_0^\infty f'(t)e^{-st}\mathrm{d}t \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\mathrm{d}t \qquad \text{(integrate by parts)} \\ &= -f(0) + sF(s)\end{aligned}$$

— provided $f(t)e^{-st} \to 0$ as $t \to \infty$

$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$ — this is how we account for I.C.'s

Similarly:

$$\begin{aligned}\mathscr{L}\{f''(t)\} = \mathscr{L}\{(f'(t))'\} &= s\mathscr{L}\{f'(t)\} - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

Example

Consider the system

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \dot{y}(0) = 0$$

(need two I.C.'s for 2nd-order ODE's)

Let's compute the transfer function: $H(s) = \dfrac{Y(s)}{U(s)}$

— take Laplace transform of both sides (zero I.C.'s):

$$s^2Y(s) + 3sY(s) + 2Y(s) = U(s) \qquad H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3s + 2}$$

Example (continued)

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \alpha,\ \dot{y}(0) = \beta$$

Compute the *step response*, i.e., response to $u(t) = 1(t)$

Caution!! $Y(s) = H(s)U(s)$ no longer holds if $\alpha \neq 0$ or $\beta \neq 0$

Again, take Laplace transforms of both sides, mind the I.C.'s:

$$s^2Y(s) - s\alpha - \beta + 3sY(s) - 3\alpha + 2Y(s) = U(s)$$

$U(s) = \mathscr{L}\{1(t)\} = 1/s$, which gives

$$s^2Y(s) - s\alpha - \beta + 3sY(s) - 3\alpha + 2Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{\alpha s + (3\alpha + \beta) + \frac{1}{s}}{s^2 + 3s + 2} = \frac{\alpha s^2 + (3\alpha + \beta)s + 1}{s(s+1)(s+2)}$$

Note: if $\alpha = \beta = 0$, then $Y(s) = \dfrac{1}{s(s+1)(s+2)} = H(s)U(s)$

Example (continued)

Compute the step response of

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \alpha,\ \dot{y}(0) = \beta$$

$$Y(s) = \frac{\alpha s^2 + (3\alpha + \beta)s + 1}{s(s+1)(s+2)} \qquad y(t) = \mathscr{L}^{-1}\{Y(s)\}$$

Use the method of partial fractions:

$$\frac{\alpha s^2 + (3\alpha + \beta)s + 1}{s(s+1)(s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2}$$

— this gives $a = 1/2,\ b = 2\alpha + \beta - 1,\ c = -\alpha - \beta + 1/2$

$$Y(s) = \frac{1}{2s} + (2\alpha + \beta - 1)\frac{1}{s+1} + \frac{-\alpha - \beta + 1/2}{s+2}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

Example (continued)

The step response of

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \alpha,\ \dot{y}(0) = \beta$$

is given by

$$y(t) = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

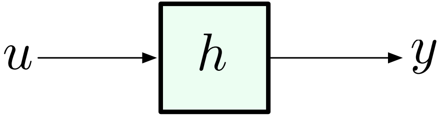
What are the transient and the steady-state terms?

- The transient terms are e^{-t}, e^{-2t} (decay to zero at exponential rates -1 and -2)

  Note the poles of $H(s) = \dfrac{1}{(s+1)(s+2)}$ at $s = -1$ and $s = -2$

  — these are *stable poles* (both lie in LHP)
- the steady-state part is $\frac{1}{2}1(t)$ — converges to steady-state value of $1/2$

DC Gain



Definition: the steady-state value of the step response is called the *DC gain* of the system.

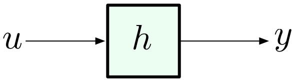
$$\text{DC gain} = y(\infty) = \lim_{t \to \infty} y(t) \qquad \text{for } u(t) = 1(t)$$

In our example above, the step response is

$$y(t) = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

therefore, DC gain $= y(\infty) = 1/2$

Steady-State Value



$$u(t) = 1(t) \qquad U(s) = \frac{1}{s} \qquad \Longrightarrow \qquad Y(s) = \frac{H(s)}{s}$$

— can we compute $y(\infty)$ from $Y(s)$?

Let's look at some examples:

- $Y(s) = \dfrac{1}{s+a}, \ a > 0$ (pole at $s = -a < 0$)
  $y(t) = e^{-at} \quad \Longrightarrow \quad y(\infty) = 0$
- $Y(s) = \dfrac{1}{s+a}, \ a < 0$ (pole at $s = -a > 0$)
  $y(t) = e^{-at} \quad \Longrightarrow \quad y(\infty) = \infty$
- $Y(s) = \dfrac{1}{s^2+\omega^2}, \ \omega \in \mathbb{R}$ (poles at $s = \pm j\omega$, purely imaginary)
  $y(t) = \sin(\omega t) \quad \Longrightarrow \quad y(\infty)$ does not exist
- $Y(s) = \dfrac{c}{s}$ (pole at the origin, $s = 0$)
  $y(t) = c1(t) \quad \Longrightarrow \quad y(\infty) = c$

The Final Value Theorem

We can now deduce the Final Value Theorem (FVT):

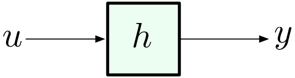
If all poles of $sY(s)$ are *strictly stable* or lie in the *open left half-plane* (OLHP), i.e., have $\text{Re}(s) < 0$, then

$$y(\infty) = \lim_{s \to 0} sY(s).$$

In our examples, multiply $Y(s)$ by s, check poles:

- $Y(s) = \dfrac{1}{s+a} \qquad sY(s) = \dfrac{s}{s+a}$
  if $a > 0$, then $y(\infty) = 0$; if $a < 0$, FVT does not give correct answer
- $Y(s) = \dfrac{1}{s^2+\omega^2} \qquad sY(s) = \dfrac{s}{s^2+\omega^2}$
  poles are purely imaginary (not in OLHP), FVT does not give correct answer
- $Y(s) = \dfrac{c}{s} \qquad sY(s) = c$
  poles at infinity, so $y(\infty) = c$ – FVT gives correct answer

Back to DC Gain



Step response: $$Y(s) = \frac{H(s)}{s}$$

— if all poles of $sY(s) = H(s)$ are strictly stable, then

$$y(\infty) = \lim_{s \to 0} H(s)$$

by the FVT.

Example: compute DC gain of the system with transfer function

$$H(s) = \frac{s^2 + 5s + 3}{s^3 + 4s + 2s + 5}$$

All poles of $H(s)$ are strictly stable (we will see this later using the *Routh–Hurwitz criterion*), so

$$y(\infty) = H(s)\Big|_{s=0} = \frac{3}{5}.$$