# Solutions to linear time-invariant state space equations

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In contrast to what we have done so far in ECE486 so far, now we switch gears and study systems in state-space form. You will often hear it be said that the state-space form is much more flexible/general. Why is the state-space approach more flexible? Very generally speaking, at the heart of any dynamical system are differential equations (or systems of equations thereof), and pretty much anything on earth changes with time (i.e. is a dynamical system). Thus, by focusing our study on the structure and properties of such systems, (1) one is able to encapsulate far more complicated systems and (2) gain more insights than possible by an input-output (or transfer function) approach.

To be more precise, a system of the form $\dot{x} = f(x)$, where $x$ denotes the different *states* of the system, allows $f(x)$ to be a wide variety of functions, including nonlinear, hybrid, switched systems, etc. Further, the variable of interest to the investigator, i.e. outputs, can be represented as a set of equations $y = h(x)$. For example, the traditional mass-spring-damper characterized by $m\ddot{x} + c\dot{x} + kx = u$ simply becomes,

$$\dot{x} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{-k}{m} & \dfrac{-c}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{1}$$

Of course, one can put (1) in the *proper* state-space form by setting $x_1 = x, x_2 = \dot{x}$. In this case, we chose the position of the mass to be our variable of interest (the output equation). While (1) is a *linear* system, nothing precludes $f(x)$ from being as arbitrarily complicated as you want. Nevertheless, for the remainder of these notes, and as we do in ECE 486, we will stick to studying linear systems. If you are interested in non-linear systems check out ECE 517/528.

As we already remarked at the beginning of the course, almost every system we examine in this course can be put in the form

$$\begin{aligned} \dot{x} &= Ax + Bu \qquad & x \in \mathbb{R}^n, u \in \mathbb{R}^m, B \in \mathbb{R}^{n \times m} \\ y &= Cx, & y \in \mathbb{R}^p, C \in \mathbb{R}^{p \times n} \end{aligned} \tag{2}$$

Such systems are called linear time-invariant (LTI) systems and when $p = m = 1$ they are called Single-Input-Single-Output (SISO) systems and MIMO systems otherwise (Multiple-Input-Multiple-Output).

Why focus on linear systems in the course? Because any nonlinear system can be *linearized* about an equilibrium state (as we discussed in the first few weeks of the course), or nominal trajectory, and in most cases of interest, the study of the linearized system is sufficient to make far-reaching conclusions about the original system itself. In the remaining few weeks, we study the concepts of *controllability, observability, stability* for such systems.

An important remark to be made is that the independent variable, $t$, is considered to belong to a dense set for the most part, i.e. $t \in \mathbb{R}$. Instead, if $t \in \mathbb{Z}$, then we have a *discrete* time system, or discrete dynamical system. We focus mostly on continuous time dynamics, and remark on the corresponding discrete time results only sporadically.

# 1 Solutions to LTI Systems

We crystallize some notation: if we write $\dot{x} = f(x)$, we mean a *autonomous uncontrolled system.* If we write $\dot{x} = f(t, x)$, we mean a non-autonomous system, i.e. its state equations depend explicitly on time, $t$. If we write $\dot{x} = f(x, u)$, we mean *controlled* system, i.e. we have some influence of the states of the system via the control variable $u$. The meaning of $\dot{x} = f(t, x, u)$ follows, and by $f(\cdot)$, we mean any type of system.

For the remainder of the notes we will deal with systems of the form (2). First let us start with a simple case of $n = 1$ and no input $u = 0$. For simplicity assume $C$ is the identity so that $y = x$. Then what is the solution $y(t) = x(t)$?

## 1.1 Scalar case

In this case (2) reduces to:

$$\dot{x} = ax, \qquad y = x$$

which is a scalar differential equation. Then, we can rewrite as

$$\frac{d}{dt}x = ax \qquad \Longrightarrow \qquad \frac{dx}{x} = adt.$$

Integrating both sides from some $t_0$ (start time) to $t$ (current time) we get:

$$\ln x = a(t - t_0) \qquad \Longrightarrow \qquad x(t) = e^{a(t-t_0)}$$

and if the initial condition at time $t = t_0$ is $x(t_0) = x_0$ then the complete solution becomes,

$$x(t) = e^{a(t-t_0)}x_0 = y(t)$$

**Exercise 1.1.** *Confirm that* $x(t) = e^{a(t-t_0)}x_0$ *is the solution to* $\dot{x} = ax$ *by differentiation.*

Now suppose we ratchet things up a notch and allow nonzero control in this scalar case. What is the solution then to the following equation?

$$\dot{x} = ax + bu \tag{3}$$

How do we solve this one? Inspired by the solution we found, let us left multiply both sides of the equation by $e^{-at}$ and proceed with some rearrangements:

$$\begin{aligned} e^{-at}\dot{x} &= e^{-at}ax + e^{-at}bu \\ e^{-at}\dot{x} - e^{-at}ax &= e^{-at}bu \end{aligned}$$

The left hand side can be written as:

$$\frac{d}{dt}\left[e^{-at}x\right] = e^{-at}bu$$

Keeping in mind that $u = u(t)$ is a function of time, integrate both sides from the initial time $t_0$ to $t$:

$$e^{-at}x(t) - e^{-at_0}x_0 = \int_{t_0}^{t} e^{-as}bu(s)ds$$

Re-arrange to get the **variation-of-constants** formula for scalar LTI systems:

$$x(t) = e^{a(t-t_0)}x_0 + \int_{t_0}^{t} e^{a(t-s)}bu(s)ds \tag{4}$$

**Exercise 1.2.** *Confirm that (4) is a solution to (3) by differentiation.* ***Hint:*** *Carefully use the Leibniz rule.*

## 1.2 General Case

Now, the pertinent question is: how do we deal with the case of the matrix equation

$$\dot{x} = Ax + Bu$$

As before, let us start with the case where the control term is identically zero, i.e., $u(t) \equiv 0$. Then our equation reduces to

$$\dot{x} = Ax \tag{5}$$

Inspired by the solution in the scalar case, let us ask: is it possible that the the solution to the system could take the following form

$$x(t) = \phi(t)x(t_0)$$

where $\phi(t)$ is some function? If it were to take such form, what properties should $\phi(t)$ have? Could we be lucky enough to get something like $e^{At}$ as before?

To start, suppose $\phi(t)$ admits a Taylor expansion in $t$.

$$\phi(t) = 1 + \phi_1 t + \phi_2 t^2 + \cdots + \phi_n t^n + \ldots$$

This assumption is in no way justified at the outset except maybe the vague hope that $\phi$ is a *nice function* (technically an analytic function).

Since $\phi(t)x_0$ is a solution of the differential equation we must have that

$$\frac{d}{dt}\left(\phi(t)x_0\right) = A\phi(t)x_0$$

This equation should also hold when $\phi(t)$ is replaced by its Taylor expansion. Thus we get,

$$\left(\phi_1 + 2\phi_2 t + \ldots + n\phi_n t^{n-1} + \ldots\right)x_0 = A\left(1 + \phi_1 t + \phi_2 t^2 + \ldots + \phi_n t^n + \ldots\right)x_0$$

Examining the above equation it becomes clear that 1 should be replaced with an identity matrix of the appropriate size. Moreover, it turns out that $\phi$ is a *matrix function*!

Collecting the coefficients we get:

$$\begin{aligned}
\phi_1 &= A \\
\phi_2 &= \frac{1}{2}A\phi_1 = \frac{1}{2!}A^2 \\
\phi_3 &= \frac{1}{3}A\phi_2 = \frac{1}{3!}A^3 \\
&\vdots \\
\phi_n &= \frac{1}{n!}A^n
\end{aligned}$$

which looks very much like the terms of the exponential function except the square matrix $A$ taking the position of the scalar $a$!

### 1.2.1 Functions of square matrices

So, we must investigate functions of square matrices. Let $f(z)$ be an analytic function defined on $z \in \mathbb{C}$. Can we make sense of $f(A)$, where $A$ is a square matrix?

Suppose $f$ is a polynomial. Then, for two invertible square matrices $V, D$, note that we have

$$f(VDV^{-1}) = Vf(D)V^{-1}$$

which follows from

$$(VDV^{-1})^k = VDV^{-1}VDV^{-1}\ldots VDV^{-1} = VD^kV^{-1}$$

In addition, if $D$ is a diagonal matrix, then $f(D)$ is simply $f$ applied to each element on the diagonal. If $f$ is not a polynomial, then consider the Taylor expansion of $f(z) = c_0 + c_1 z + c_2 z^2 + \dots$, and replace each power of $z$ with a power of $A$. The Cayley-Hamilton Theorem states that every square matrix satisfies its characteristic polynomial. Therefore, it is clear that we only need to compute a finite number of powers of $A$. If $A$ is diagonalizable, or has $n$ distinct eigenvalues, then we have managed to define $f(A)$ for a wide class of functions $f(z)$; in particular if $f(z)$ converges for $|z| < r$ then $f(A)$ will converge for $\|A\| < r$, where $\|\cdot\|$ is a *matrix norm.* Note that, in this set up, we cannot consider expansions about matrices, i.e., an expansion of $f(A + \eta B)$ about $\eta = 0$ for example will prove troublesome, unless $A$ and $B$ commute.

Now, consider the Jordan Normal Form of a matrix. All complex matrices admit a Jordan normal form $A = PJP^{-1}$, where $J$ consists of Jordan blocks. By above observations, we can extend $f(A)$ to non-diagonalizable matrices as long as we can compute $f(J)$. Applying the Taylor expansion to an $n \times n$ Jordan block results in,

$$f\left(\begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & \lambda & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{bmatrix}\right) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} & \dots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & f(\lambda) & f'(\lambda) & \vdots & \frac{f^{(n-2)}\lambda}{(n-2)!} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & f(\lambda) & f'(\lambda) \\ 0 & \dots & \dots & 0 & f(\lambda) \end{bmatrix} \tag{6}$$

We note that, for a repeated eigenvalues, the number of Jordan blocks is determined by the number of independent eigenvectors corresponding to it. Alternatively, if a Jordan block for a $\lambda$ has size 2 or more, it corresponds to an eigenvector deficit.

**Take-away:** All this to say that analytic functions of square matrices are well defined and in particular, the **matrix exponential** is a well defined quantity for all square matrices with real or complex entries.

### 1.2.2 Solution to matrix differential equation $\dot{x} = Ax$

Now, recall that the solution to $\dot{x} = ax$ is given by $x = e^{at}x_0$ in the single variable case (i.e. $n = 1$), where $x_0$ is the initial condition. Now, having defined functions of a square matrix, we can define $e^{At}$ via the power series:

$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

and claim the solution to $\dot{x} = Ax$ is given by $e^{At}x_0$. The convergence of $e^A$, for any square matrix $A$, follows from (6), since each element of the matrix converges (because $e^\lambda = \sum\limits_{k=0}^{\infty} \dfrac{\lambda^k}{k!}$ converges for every $\lambda$). Then, $e^{At}$ converges to a matrix for every finite $t$.

It remains to show $e^{At}x_0$ is a solution to $\dot{x} = Ax$. Note $x\left(0\right) = e^{A\cdot 0}x_0 = Ix_0 = x_0$ and moreover

$$\frac{d}{dt}e^{At} = A + tA^2 + \ldots = Ae^{At} \Rightarrow \dot{x} = \frac{d}{dt}\left(e^{At}x_0\right) = A\left(e^{At}x_0\right) = Ax$$

Therefore, $e^{At}$ is a solution as claimed.

### 1.3 General case (again)

Now let us go back to the original system we wanted to solve:

$$\dot{x} = Ax + Bu,$$

Let us assume a solution of the form

$$x(t) = e^{At}c(t).$$

If we differentiate this assumed solution we get:

$$\dot{x}(t) = Ae^{At}c(t) + e^{At}\dot{c}(t) = Ax + e^{At}\dot{c}(t).$$

Matching $\dot{x} = Ax + Bu$ gives:

$$e^{At}\dot{c}(t) = Bu \qquad \Longrightarrow \qquad \dot{c}(t) = e^{-At}Bu.$$

Integrating:

$$c(t) = x(t_0) + \int_{t_0}^{t} e^{-As}Bu(s)\,ds.$$

Thus we get that the final solution (variation of constants formula) is:

$$x(t) = e^{At}x_0 + \int_{t_0}^{t} e^{A(t-s)}Bu(s)\,ds \tag{7}$$

**Exercise 1.3.** *Confirm that (7) solves (2).* ***Hint:*** *Carefully use the Leibniz rule.*

### 1.4 Computing matrix exponentials

So for matrix equations of the form $\dot{x} = Ax$ (i.e. absent the control term), we need to compute matrix exponentials. However, what we have above is an infinite series. How do we compute $e^{At}$ in practice? It turns out this is where a little bit of linear algebra will come in handy.Let us look at a few examples to get a feel for the computations involved.

### 1.4.1 Example: Diagonal matrices

Suppose

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Then we have,

$$A^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \quad \Rightarrow \quad e^{At} = I + \begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix} + \ldots + \begin{bmatrix} \lambda_1^k t^k / k! & 0 \\ 0 & \lambda_2^k t^k / k! \end{bmatrix} + \ldots$$

This is simply,

$$e^{At} = \begin{bmatrix} 1 + \lambda_1 t + \frac{\lambda_1{}^2}{2!} t^2 + \ldots & 0 \\ 0 & 1 + \lambda_2 t + \frac{\lambda_2^2}{2} t^2 + \ldots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 \tau} \end{bmatrix}$$

**Note:** This should make sense because in the diagonal case the equations are actually decoupled.

### 1.4.2 Example: Upper shift matrices

Now consider the case when

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We easily observe $A^2 = 0$ and therefore,

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

What if now:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrices with zeros everywhere except *superdiagonal* are called **shift matrices**.

What is $e^{At}$ now? The key observation is that in this case $A^3 = 0$. We leave it as an exercise to the reader to compute $e^{At}$ in this case. The general form of the matrix exponential for shift matrices is:

$$\exp\left( \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \\ 0 & \ldots & & 0 & 0 \end{bmatrix} t \right) = \begin{bmatrix} 1 & t & t^2/2 & \ldots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ & & 1 & \ddots & t^2/2 \\ \vdots & & & \ddots & t \\ 0 & \ldots & & 0 & 1 \end{bmatrix} \tag{8}$$

**Remark:** We say a matrix $N$ is nilpotent if $N^k$=0 for some $k > 0$. Any such $N$ is similar to a block diagonal matrix where all the diagonal blocks are shift matrices (possibly of different sizes).

### 1.4.3 Example: Jordan Blocks

Now suppose,

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

We immediately observe that $A = A_1 + A_2$ where

$$A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and we have already discussed the case of $e^{A_1 t}$ and $e^{A_2 t}$. Is $e^{At} = e^{(A_1+A_2)t}$ related to $e^{A_1 t}$ and $e^{A_2 t}$? We know it is true for scalars that $e^{a+b} = e^a e^b$. Does it remain true for the matrix exponential as we have painstakingly defined above?

**Exercise 1.4.** *Let $A_1$ and $A_2$ be arbitrary square matrices (not necessarily the ones defined above). Show that*

$$e^{A_1+A_2} = e^{A_1} e^{A_2}$$

*only if $A_1$ and $A_2$ commute. That is $A_1 A_2 = A_2 A_1$.*

It turns out that matrices don't quite behave like scalars with respect to multiplication and so $e^{A_1} e^{A_2} \neq e^{A_1+A_2}$ with commutativity of $A_1$ and $A_2$ being the stumbling block.

Fortunately for us, one of our $A_k$ is a multiple of the identity matrix, and that commutes with with everything. Therefore we get:

$$e^{At} = e^{A_1 t} e^{A_2 t} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

Moreover, for a general Jordan Block we have that:

$$A = \begin{bmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda \end{bmatrix} \qquad \Rightarrow \qquad e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \dots & \dfrac{t^{n-1} e^{\lambda t}}{(n-1)!} \\ 0 & e^{\lambda t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & te^{\lambda t} \\ 0 & \dots & 0 & e^{\lambda t} \end{bmatrix}$$

### 1.4.4 Example: Complex blocks

The last matrix we will consider is:

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

You may know from linear algebra (MATH257 maybe, MATH416 certainly) that such matrices arise in the *Real Jordan Form* (see Section 2 of this note). Now we see that we can once again write $A = A_1 + A_2$ if we define,

$$A_1 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

enabling us to once again repeat the trick we performed above; **provided** we know how to compute $e^{A_2 t}$.

**Exercise 1.5.** *Compute a few values of* $A_2^k$*, e.g for* $k = 2, 3, 4, 5$*. Show then the summation of these powers below has the following form:*

$$\sum_{k=0} A_2^k = \begin{bmatrix} 1 - b^2 + b^4 - b^6 + \dots & -b + b^3 - b^5 + \dots \\ b - b^3 + b^5 + \dots & 1 - b^2 + b^4 - b^6 + \dots \end{bmatrix}$$

*Do you recognize the entries of the matrix above?*

Based on the result of the above exercise we see that $e^{A_2 t}$ results in a rotation matrix and thus:

$$e^{At} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix} \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}$$

## 1.5 Summary

From linear algebra, we know that *any* matrix $A$ can be reduced to a canonical form $\bar{A}$ which is comprised of one of the types of blocks from the examples we have discussed above via a similarity transformation $\bar{A} = P^{-1}AP$.

Thus, now we are in position to have defined the matrix exponential for all matrices ***provided*** we know how the similarity transformation itself is changed under exponentiation.

For this, note:

$$\bar{A} = P^{-1}AP \quad \Rightarrow \quad A = P\bar{A}P^{-1}$$

and we know how to compute $e^{\bar{A}t}$ where $\bar{A}$ is comprised of canonical blocks.

Plug-in $A$ into the definition of the matrix exponential:

$$\begin{aligned} e^{At} &= I + At + \frac{A^2t^2}{2} + \dots = I + P\bar{A}P^{-1}t + \frac{P\bar{A}P^{-1}P\bar{A}P^{-1}t^2}{2} + \dots \\ &= P\left(I + \bar{A} + \frac{\bar{A}^2t^2}{2} + \dots\right)P^{-1} = Pe^{\bar{A}t}P^{-1} \end{aligned}$$

which finally provides us with an algorithm for computing $e^{At}$.

1. Convert $A$ to the canonical form $\bar{A}$ and store the similarity transform matrix $P$.
2. Compute $e^{\bar{A}t}$ based on our knowledge so far.
3. Finally $e^{At} = Pe^{\bar{A}t}P^{-1}$.

# 2 Diagonalization

In the first section of this note, we relied quite heavily (by fiat) on the fact that there exist similarity transformations that put square matrices in diagonal or nearly diagonal form. The purpose of this section is to review this basic fact. Recall that we say two matrices $A$ and $\bar{A}$ are related by a **similarity transformation** if $\bar{A} = PAP^{-1}$ for some invertible matrix $P$ called a change-of-basis matrix.

From our experience with physics and engineering, we are familiar with situations where a change of coordinates makes the problem more straightforward or more amenable to analysis (e.g., spherical & cylindrical coordinates). One should think of the change of basis matrix $P$ as linear transformation that facilitates such a change of coordinates. One critical such case is when we express an operator or transformation in terms of its **invariants**. A few definitions now follow to make this statement more precise in the context of linear operators and their matrix representations.

## 2.1 Eigenvectors & eigenvalues

Given a square matrix $A$, we say a vector $v$ is an **eigenvector** of $A$ if

$$A(v) = \lambda v, \quad \lambda \in \mathbb{C}$$

In other words the operator only scales the vector $v$ in magnitude. We say the scaling factor $\lambda$ is an **eigenvalue**.

In terms of the matrix representation, we thus get that an eigenvector $v$ belongs to the kernel of the operator $(A - \lambda I)$:

$$Av = \lambda v \quad \Leftrightarrow \quad (A - \lambda I)\, v = 0$$

Moreover, we see that the span of any given eigenvector is *invariant* under the action of $A$. (Recall the span of a vector $v$ is $\alpha v$ where $\alpha \in \mathbb{C}$).

## 2.2 Characteristic polynomial

Given a square matrix $A$, its **characteristic polynomial** is defined as the monic polynomial (in variable $\lambda$) given by $\det(A - \lambda I)$. The roots of the characteristic polynomial are thus the *eigenvalues* of $A$.

**Exercise 2.1.** *Show that the characteristic polynomial (and thus the eigenvalues) are invariant under similarity transformations.*

**Note:**

1. While eigenvalues are invariant under similarity transforms, eigenvectors are **not**.
2. An $n \times n$ square matrix $A$ need not have $n$ distinct eigenvalues (Why? Hint: Polynomials can have repeated roots).

When $A$ has $n$ **distinct** eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, it *also* has $n$ eigenvectors $\{v_1, \ldots, v_n\}$ (it need not if $\lambda_i$ are not distinct, a case we will discuss a little while later).

**Key Point:** It is a standard fact from linear algebra that in the basis formed by these $n$ eigenvectors. $\{v_1, \ldots, v_n\}$, $A$ becomes a *diagonal matrix.*

$$A = VDV^{-1}$$

## 2.3 Regular matrices

**Example 2.1.** Let us do a partial example (i.e., you should fill in the steps if any are missing). Let us find the *eigensystem* (or all eigenvalues & eigenvectors) of

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and verify in the *eigencoordinates* the matrix is diagonalized.

**Solution:** Considering the determinant of $A - \lambda I$ we have,

$$\det(A - \lambda I) = (\lambda - 1)(\lambda - 2)^2 \qquad \Longrightarrow \qquad \lambda_i = \{1, 2, 2\}$$

which determines the eigenvalues. Let us plug these values and evaluate the equation(s):

$$\left.\begin{bmatrix} 1-\lambda & 3 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right|_{\lambda_i = 1,2,2} \cdot \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = 0$$

***Note*:** We are using superscripts to refer to the components of the vector since we will use subscripts ($v_1, v_2$ and $v_3$) to denote the eigenvectors themselves.

We get for $\lambda_1 = 1$,

$$\begin{bmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = 0 \qquad \Longrightarrow \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We see that the equations specify $v^2 = v^3 = 0$ and $v^1$ is free to be anything (we chose 1 arbitrarily).

Now for $\lambda = 2$,

$$\begin{bmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = 0 \tag{9}$$

Here we see that $v^2$ and $v^3$ are free to be anything as long as the equation:

$$-v^1 + 3v^2 + v^3 = 0$$

is satisfied. This gives us two degrees of freedom:

- First set $v^3 = 0$ giving $3v^2 = v^1$.
- Next set $v^2 = 0$ giving $v^1 = v^3$.

Thus we get

$$\begin{bmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = 0 \quad \implies v_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as two eigenvectors for each of the eigenvalues $\lambda_2 = \lambda_3 = 2$.

Now form the change of basis matrix using by collecting $v_k$ as the columns of $P$:

$$P = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify that

$$D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

## 2.4 Defective matrices

In preparation for next section, consider the slightly altered matrix:

$$E = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

and repeat the example. What happened?

### 2.4.1 Jordan normal form

Matrices like $E$ above are called *defective* matrices. What happens in the case of $E$ is that we are unable to find a second eigenvector for the repeated eigenvalue $\lambda = 2$. The two equations we get are:

$$\begin{bmatrix} 0 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} -1 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = 0 \tag{10}$$

The first gave the same eigenvector $v_1$ as before, however, from the second equation we see that we only get the $v_2$ from before but not $v_3$ (why?). Thus, in this case we are not able to form a change of basis matrix $P$ which is invertible. It isn't completely obvious that we cannot find *some* transformation or basis-change in which the matrix is transformed to a diagonal one.

**How should we proceed?**

While, it is actually true that we cannot diagonalize $E$, we can "nearly" diagonalize it. Let us note one essential difference, the matrices corresponding to $\lambda = 2$ in (10) and (9) have different *nullity*!

Recall from MATH257 (or another linear algebra course) that by *nullity* of some matrix $M$ we mean the dimension of the nullspace of $M$.

In particular, we have, $\dim(N(A-2\cdot I)) = 2$ while $\dim(N(E-2\cdot I)) = 1$, which explains why we cannot find a second eigenvector.

Recall that eigenvector for $A$ lies in the nullspace of the operator $(A - \lambda I)$ and that the nullspace of a matrix is an invariant subspace.

For the matrix $E$ we have found a basis for $(E - 2I)$ in line with the nullity of that matrix, but that isn't quite enough to diagonalize $E$ via a similarity transformation. What is the next best thing?

Is it possible there is some $k > 0$ and $v$ such that,

$$(E-2I)^{k-1}\, v \neq 0 \qquad \text{but} \qquad (E-2I)^{k}\, v = 0$$

**Intuition:** It might seem like that the above condition came out of nowhere, but we can, (loosely) interpret the above condition as looking for vectors $v$, which under the action of $(E - \lambda I)$ are a "few iterations away" from belonging to its kernel. Or more explicitly, note that for $\lambda = 2$

$$(E-2I)^{k}\, v = (E-2I)\,(E-2I)^{k-1}\, v = 0$$

implies $y := (E-2I)^{k-1}\, v$ is then in the kernel of the characteristic matrix.

We call such a vector $v$ a generalized eigenvector of rank/grade $k$. Note that, when $k = 1$ we see that we recover the notion of standard eigenvector.

**Definition 2.1.** For a linear operator $A$, we say $v$ is a **generalized eigenvector** of grade $k$ corresponding to eigenvalue $\lambda$, if for some $k > 0$ we have that,

$$(A-\lambda I)^{k-1}\, v \neq 0 \qquad \text{but} \qquad (A-\lambda I)^{k}\, v = 0$$

It turns out that by using the generalized eigenvector(s) $v$ to complete a basis for $P$ we get a similarity transformation that puts $A$ in a *nearly diagonal form.*

**Exercise 2.2.** *Verify that:*

1. *The vector* $v_3 = \begin{bmatrix}-2 & 0 & 1\end{bmatrix}^T$ *is a generalized eigenvector of grade 2 of* $E$ *above corresponding to* $\lambda = 2$*.*

2. *The matrix $P$ formed by using adopting $v_1, v_2, v_3$ as its columns satisfies:*

$$P^{-1}EP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The “nearly diagonal form” that we get by using generalized eigenvectors has a special name: **Jordan Normal Form**. The Jordan form matrix is composed of what we call *Jordan Blocks* of different sizes along the diagonal.

**Definition 2.2. Jordan Blocks** are square matrices of the form below:

$$J^1[\lambda] := \lambda, \quad J^2[\lambda] := \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad J^3[\lambda] := \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad \ldots$$

where the superscript (optionally) identifies the size of the block.

The matrix specified as $P^{-1}EP$ above consists of two Jordan blocks. The main point of this section of the notes is the following fact:

Every linear operator (a.k.a square matrix) $M$ with real or complex entries, is similar to a block-diagonal matrix:

$$J = \begin{bmatrix} J^{n_1} & & \\ & \ddots & \\ & & J^{n_p} \end{bmatrix}$$

where the blocks $J^{n_i}$ are of the form

$$J^{n_i} = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}_{n_i \times n_i}$$

and the **geometric** multiplicity of some $\lambda_k$ is the number of Jordan blocks $J^{n_i}$ having $\lambda_k$ on the diagonal while the **algebraic multiplicity** of $\lambda_k$ is the sum of the *sizes* $\{n_i\}$ of the Jordan blocks having $\lambda_k$ on the diagonal.

**Exercise 2.3.** *Find the similarity transformation that puts $A$ in Jordan Form:*

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -3 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

1. *Over the complexes*

2. *Over the reals (i.e. avoid use of complex numbers)*

### 2.4.2 Real Jordan Form

Given a square matrix $A$ with **only** real valued entries, it is possible for the matrix to have complex valued eigenvectors. Correspondingly, when diagonalized, it often becomes the case that the matrix $D$ suddenly has complex valued entries. There is something unsatisfactory about this situation – we started with a purely real matrix, diagonalized it, and lo and behold the decomposition suddenly involves complex numbers!

Fortunately, for us, this situation can be resolved by splitting the complex valued eigenvectors into their real and imaginary parts and by writing the complex eigenvalued Jordan Block in a particular fashion. We illustrate this with an example:

**Example 2.2.** Consider the following diagonal matrix:

$$\begin{bmatrix} a+bi & \\ 0 & a-bi \end{bmatrix}$$

It is obvious that the eigenvalues of the matrix above are $a \pm bi$. Now consider the following two matrices:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

One can readily verify that the eigenvalues of the above two matrices are also $a \pm bi$. Thus, we can replace a complex Jordan Block $J^1[\lambda]$ and its conjugate partner $\overline{J}^1[\lambda]$ with a single **real Jordan Block** constructed out of the real and imaginary parts of the eigenvalues.

While we have resolved the problem of complex valued entries in the diagonalized matrix, we still have not addressed how to deal with complex entries in the eigenvectors (from which the coordinate change that performs diagonalization is constructed). It turns out that the same trick of separating the real and imaginary parts of the complex numbers still works when we use it on eigenvectors. We leave it to the reader to explore the mathematics that allows this but show the result with an exercise.

**Exercise 2.4.** *Consider the matrix $A$ below:*

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

1. *Show that the relation $A = P^{-1}DP$ where*

$$D = \begin{bmatrix} 1+2i & 0 & 0 \\ 0 & 1-2i & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad P = \begin{bmatrix} i & -i & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   *diagonalizes the matrix.*

*2. Show that the same is true for the relation* $A = \bar{P}^{-1}\bar{D}\bar{P}$ *where*

$$\overline{D} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \overline{P} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Note: The columns of* $\bar{P}$ *are constructed from the columns of* $P$ *by adding and subtracting the complex eigenvector and its conjugate and dividing by 2.*

## 3 Stability

Now we turn to the question of stability. In class, we briefly made the following statement:

> A system $\dot{x} = Ax + Bu$ is open-loop stable (i.e., no feedback applied) if all eigenvalues of $A$ are in the open left half plane.

While the above statement is true, it is not quite the complete picture. The purpose of this last section is to provide a more precise statement.

Given a system $\dot{x} = f(x)$, we define its equilibrium to be the set $\{x_e : f(x_e) = 0\}$. It is clear that this corresponds to the fixed points of the system. A natural question to ask then, is if this fixed point is *stable*. What do we mean by that? Simply, that if the system starts at this fixed point, how does it behave under slight perturbations from that fixed point. The classical example is the pendulum, which has two fixed points; intuitively one is stable while the other is not. A mathematically precise statement of the above notion can be written as follows:

**Definition 3.1** (Stability)**.**

1. A fixed point, $x_e$, of the system, $\dot{x} = f(x)$, is said to be stable if $\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0$ such that if $|x_0 - x_e| \leq \delta_\varepsilon$ then $|x - x_e| \leq \varepsilon$ for all $t$.
2. A fixed point $x_e$ is said to be asymptotically stable if it is possible to choose $\delta_\varepsilon$ in (1) such that $x \to x_e$ as $t \to \infty$.
3. A fixed point is said to be globally asymptotically stable if $\delta$ in (2) can be chosen independent of the given $\varepsilon$.

If the system under consideration is non-autonomous, then these definitions need a refinement to account for the explicit dependence on time, which is beyond the scope of these notes.

As important as these definitions are, they are not useful in investigating the stability of a given fixed point $x_e$. Indeed, for a general system $\dot{x} = f(x)$, the question of stability has a deep and rich theory beyond our scope. Instead, for now we focus on linear systems i.e. $\dot{x} = Ax$, where $A \in \mathbb{R}^{n \times n}$. By doing so, we can employ tools from linear algebra to results of interest. Before we state the theorem some notation is in order. Let $A$

be a square matrix. Then by $\sigma(A)$ we denote the **spectrum** of $A$, i.e. the set of its eigenvalues:

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid Av = \lambda v \text{ for some nonzero vector } v\}$$

Given a complex number $\lambda$ we denote by $\Re(\lambda)$ and $\Im(\lambda)$ its real and imaginary parts. A complete characterization of open-loop stability of a system is the following theorem:

**Theorem 3.1** (Uncontrolled linear system stability)**.**

1. *A system* $\dot{x} = Ax$ *is asymptotically stable if and only if* $\forall\lambda \in \sigma(A), \Re(\lambda) < 0$*. It is stable if and only if* $\forall\lambda \in \sigma(A), \Re(\lambda) \leq 0$*, and for each* $\lambda$ *with multiplicity* $m$ *such that* $\Re(\lambda) = 0$*, there exist* $m$ *linearly independent eigenvectors.*

2. *A discrete time system* $x_k = Ax_{k-1}$ *is asymptotically stable if and only if* $|\lambda| < 1$*. It is stable if and only if* $\forall\lambda \in \sigma(A), |\lambda| \leq 1$*, and for each* $\lambda$ *with multiplicity* $m$ *such that* $|\lambda| = 1$*, there exist* $m$ *linearly independent eigenvectors.*

Now, we can state the proof of the theorem.

**Proof of Theorem 3.1** Consider $\bar{x} = Px$ a change of variables where $P$ is non-singular. Then $\dot{x} = Ax$ is equivalent to $\dot{\bar{x}} = \bar{A}x = PAP^{-1}\bar{x}$, i.e. a solution $\bar{x}(t)$ remains bounded if and only if a solution $x(t)$ remains bounded, since they are related by a change of variables. Now, similar matrices share the same eigenvalues. Therefore, it is sufficient to study $\dot{\bar{x}} = \bar{A}\bar{x}$, where $\bar{A}$ is a Jordan matrix. For the solution $e^{\bar{A}t}\bar{x}_0$, we can examine each Jordan block in $\bar{A}$. Then, $e^{J_k t}$ for a Jordan block $J_k$ becomes, by (6),

$$e^{J_k t} = \begin{bmatrix} e^{\lambda_k} & te^{\lambda_k t} & \frac{t^2 e^{\lambda_k t}}{2} & \dots & \frac{t^{n-1}e^{\lambda_k t}}{(n-1)!} \\ 0 & e^{\lambda_k} & te^{\lambda_k t} & \vdots & \frac{t^{n-2}e^{\lambda_k t}}{(n-2)} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & e^{\lambda_k} & te^{\lambda_k t} \\ 0 & \dots & \dots & 0 & e^{\lambda_k} \end{bmatrix} \bar{x}_0$$

Now, it is clear that if all $\Re(\lambda) < 0$, each term in the matrix above tends to zero, and the solution will be asymptotically stable. If any $\Re(\lambda) > 0$, then at least one component will grow unbounded. If any $\Re(\lambda) = 0$, and the corresponding Jordan block is not of order 2 or higher, then the corresponding entry in the matrix above is constant or sinusoidal and therefore bounded. Otherwise, at least one entry of the matrix above will grow unbounded. This completes the proof for the continuous time case.

In the discrete time case, the solution at time $k$ is given by $x(k) = A^k x_0$. Assume the corresponding Jordan matrix is $\Lambda^k$. Then, the diagonal terms of $\Lambda^k$ are $\lambda^k = |\lambda|^k e^{i\angle\lambda}$, where $|\cdot|$ denotes the magnitude, and $\angle\cdot$ denotes the angle of a complex number. These terms are bounded if and only if $|\lambda| \leq 1$, and they converge to zero if and only if $|\lambda| < 1$. For blocks corresponding to eigenvalues of algebraic multiplicity greater than 1, a term on the $n^{th}$ *super diagonal* of $\Lambda^k$ has form, $\begin{pmatrix} k \\ n \end{pmatrix} \lambda^{k-n}$. These terms are bounded (and converges to zero) if and only if $|\lambda| < 1$. ■