Problems:

1. Consider the nonlinear differential equation:

$$\ddot{y} = 2y - (y^2 + 1)(\dot{y} + 1) + u$$

a) Obtain a non-linear state-space representation.

b) Linearize this system of equations around its equilibrium output trajectory when $u(\cdot) \equiv 0$, and write it in state-space form.

2. Suppose $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{m\times m}$ are square matrices. Suppose A and D have all distinct eigenvalues. (That is, the eigenvalues of A are both different from each other *and* the eigenvalues of D, and similarly D.) Prove that the eigenvalues of M are the union of the eigenvalues of A and D, where:

$$M = \begin{bmatrix} A & B \\ 0_{m\times n} & D \end{bmatrix}$$

Here, $0_{m\times n} \in \mathbb{R}^{m\times n}$ is the matrix of all zeros, and $B \in \mathbb{R}^{n\times m}$ is an arbitrary matrix.
Hint: Use the eigenvectors of A and D to construct the eigenvectors of M. Note that $(sI - A)$ is invertible for any s that is *not* an eigenvalue of A.
Note: This is actually true for any A and D, but is easier to show for the distinct eigenvalue case.

3. Consider:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Suppose D is invertible. Show that $\det(M) = \det(D)\det(A - BD^{-1}C)$. (This is known as the Schur complement; note how this generalizes the 2×2 equation for the determinant: $ad - bc$.)
Hint: You may use the previous problem, and you may take it for granted that $\det(AB) = \det(A)\det(B)$. (In abstract algebra terms, this means that the determinant is a group homomorphism.) Try to break down M into the product of two triangular matrices, one with determinant $\det(D)$ and one with determinant $\det(A - BD^{-1}C)$.

4. Consider the linear system:

$$\dot{x} = Ax + Bu \qquad y = Cx \qquad x(0) = x_0 \tag{1}$$

For any time $T > 0$, we can view this system as a mapping:

$$L : (x_0, (u(t))_{0\le t\le T}) \mapsto (x_f, (y(t))_{0\le t\le T})$$

That is, L takes initial conditions $x(0) = x_0$ and functions $u(\cdot)$ as an input, and it outputs final states $x(T) = x_f$ and functions $y(\cdot)$, according to the differential equation (1). Let $\mathcal{U}$ denote the set of piecewise continuous, square-integrable functions from $[0, T]$ to $\mathbb{R}^{n_i}$, and similarly $\mathcal{Y}$ denote the set of piecewise continuous, square-integrable functions from $[0, T]$ to $\mathbb{R}^{n_o}$. So, $L : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n \times \mathcal{Y}$.

The *dual system* is given by:

$$-\dot{\tilde{x}} = A^\intercal \tilde{x} + C^\intercal \tilde{u} \qquad \tilde{y} = B^\intercal \tilde{x} \qquad \tilde{x}(T) = \tilde{x}_f$$

Here, $\tilde{u} \in \tilde{\mathcal{U}} = \mathcal{Y}$ and $\tilde{y} \in \tilde{\mathcal{Y}} = \mathcal{U}$. **Note the time index and the minus sign on the state dynamics**; we'll actually think of the dual system moving *backward* in time. Define:

$$L^* : (\tilde{x}_f, (\tilde{u}(t))_{0 \le t \le T}) \mapsto (\tilde{x}_0, (\tilde{y}(t))_{0 \le t \le T})$$

L^* maps *final* states $\tilde{x}_f$ and dual inputs $\tilde{u}$ to initial states $\tilde{x}_0$ and dual outputs $\tilde{y}$. Note that $L^* : \mathbb{R}^n \times \tilde{\mathcal{U}} \to \mathbb{R}^n \times \tilde{\mathcal{Y}}$.

Define the inner product on $\mathbb{R}^n \times \mathcal{U}$ (which is also $\mathbb{R}^n \times \tilde{\mathcal{Y}}$) as:

$$\langle (x_0, u(\cdot)), (x_0', u'(\cdot)) \rangle = x_0^\intercal x_0' + \int_0^T u(t)^\intercal u'(t) dt$$

Define the inner product on $\mathbb{R}^n \times \mathcal{Y}$ similarly.

For this problem, show that L^* is the adjoint of L. (This is sometimes called the *pairing lemma*.)

Hint: Consider $\frac{d}{dt}\langle x, \tilde{x} \rangle$, and integrate on $[0, T]$.