
Problems:

1. Let $R[x]_{\leq n}$ be the space of polynomials with real coefficients of degree at most n defined on the field of reals $\mathbb{R}$. Let $\mathcal{A} : R[x]_{\leq n} \to R[x]_{\leq n}$ be the derivative operator (e.g., $\mathcal{A}[2x^2 + 3x + 1] = 4x + 3$).

a) Show that $(R[x]_{\leq n}, \mathbb{R})$ is a vector space and $V = \{1, x, x^2, \ldots, x^n\}$ is a basis for it.

b) Show that $\mathcal{A}(\cdot)$ is a linear operator, and find the matrix representation of $\mathcal{A}$ in terms of basis V. That is, find a matrix A such that for every $f \in R[x]_{\leq n}$, $[\mathcal{A}(f)]_V = A[f]_V$, where $[g]_V \in \mathbb{R}^{n+1}$ is the representation of g with respect to the basis V.

2. Suppose $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ is a linear operator. Show that $\dim(N(\mathcal{A})) + \dim(R(\mathcal{A})) = \dim(\mathcal{X})$. This is known as the *rank-nullity theorem.*

Hint: Let $\{v_1, \ldots, v_k\}$ be a basis of $N(\mathcal{A})$. Show that this basis can be extended to a basis for $\mathcal{X}$ by adding additional independent vectors $\{v_{k+1}, \ldots, v_n\}$. Then show $A(v_{k+1}), \ldots, A(v_n)$ is a basis of $R(\mathcal{A})$.

3. As we discussed in class, every matrix can be put into Jordan form:

$$A = T \begin{bmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_k \end{bmatrix} T^{-1}$$

Here, each Jordan block J_i has the form:

$$\begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & 0 & \lambda_i \end{bmatrix}$$

a) Let J be a Jordan block of size n. Show that:

$$J^k = \begin{bmatrix} \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \cdots & \binom{k}{n-1}\lambda^{k-(n-1)} \\ & \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \cdots & \binom{k}{n-2}\lambda^{k-(n-2)} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} \\ & & & & & \binom{k}{0}\lambda^k \end{bmatrix}$$

Here, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, and we use the convention that $\binom{n}{k} = 0$ if $n - k < 0$.

Hint: You may use a proof by induction, and Pascal's rule may prove useful:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

b) If the Jordan block J_i is of size n, then for any analytic function f:

$$f(J_i) = \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \frac{f''(\lambda_i)}{2} & \dots & \frac{f^{(n-1)}(\lambda_i)}{(n-1)!} \\ 0 & f(\lambda_i) & f'(\lambda_i) & \dots & \frac{f^{(n-2)}(\lambda_i)}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & f(\lambda_i) & f'(\lambda_i) \\ 0 & 0 & 0 & 0 & f(\lambda_i) \end{bmatrix}$$

Use the result from the previous problem to prove this.

Hint: Since f is analytic, we may write $f(s) = \sum_{k=0}^{\infty} \alpha_k s^k$. Using this, write out $f(J)$, and consider what the entries in $f(J)$ will be.

Note: One consequence of what you just proved is the following:

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2}{2}e^{\lambda_i t} & \dots & \frac{t^{n-1}}{(n-1)!}e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & te^{\lambda_i t} & \dots & \frac{t^{n-2}}{(n-2)!}e^{\lambda_i t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_i t} & te^{\lambda_i t} \\ 0 & 0 & 0 & 0 & e^{\lambda_i t} \end{bmatrix}$$

Thus, by answering parts (a) and (b), you have shown that:

$$e^{At} = T \begin{bmatrix} e^{J_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{J_k t} \end{bmatrix} T^{-1}$$

4. Suppose that A and Q are $n \times n$ matrices, and consider the matrix differential equation:

$$\dot{Z} = AZ + ZA^* \qquad\qquad Z(0) = Q \tag{1}$$

a) Show using the product rule that the unique solution to (1) is given by:

$$Z(t) = e^{At} Q e^{A^* t}$$

b) Show that if $e^{At} \to 0$ as $t \to \infty$, then

$$P = \lim_{t_f \to \infty} \int_0^{t_f} Z(t)dt$$

is a solution to the *Lyapunov equation*:

$$AP + PA^* + Q = 0$$

(**Hint:** Integrate both sides of (1) from 0 to t_f and use the fundamental theorem of calculus.)