

ECE 515/ME 540 (Control System Theory and Design) – Homework 4

Due: Thursday, October 10 at 2pm

Problem 1. Use stability definitions to determine whether the following systems are Lyapunov stable (i.e. stable in the sense of Lyapunov), asymptotically stable, globally asymptotically stable, or none. The first two systems are in \mathbb{R}^2 and the last one is in \mathbb{R} .

- a) $\dot{x}_1 = 0$
 $\dot{x}_2 = -x_2$
- b) $\dot{x}_1 = -x_2$
 $\dot{x}_2 = 0$
- c) $\dot{x} = 0$ if $|x| > 1$
 $\dot{x} = -x$ if $|x| \leq 1$

Carefully justify your answers, using only the definitions of stability. (Do not use eigenvalue methods or Lyapunov's method.)

Problem 2. Prove the variation-of-constants formula for linear time-varying ordinary differential equations, stated in Section 3.7 of the course reader and repeated here:

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^t \phi(t, s)B(s)u(s)ds \quad (1)$$

Problem 3. Let M be a symmetric real-valued $n \times n$ matrix. Show that the following three statements are equivalent:

- a) M is positive definite.
- b) All eigenvalues of M are positive.
- c) $M = N^T N$ for some nonsingular $n \times n$ matrix N .

Hint: show $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

Problem 4. Suppose $V(x) = x^T A x$ for a real matrix A . Prove that if the function V is positive definite, then the matrix $M = \frac{1}{2}(A + A^T)$ is positive definite.

Provide an example of a positive definite V where the matrix A itself is not symmetric.

Problem 5. Consider the LTI system described by:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix} x$$

Investigate asymptotic stability using the Lyapunov equation:

$$A^T P + P A = -Q \quad Q = I$$

Can you arrive at any definite conclusion?

Problem 6. In this problem, we will investigate Lyapunov functions for discrete-time systems.

Consider the discrete-time system:

$$x[k+1] = f(x[k]) \quad k = 0, 1, \dots$$

- a) Suppose we have a Lyapunov function $V(x) = x^T P x$ for some symmetric, positive-definite P . What is $V(f(x)) - V(x)$? (This is analogous to $\dot{V}(x(t))$ in continuous time, which is given by $\nabla V \cdot f$.)

- b) In the LTI case where $f(x) = Ax$, show that the origin is asymptotically stable if for some $Q \succ 0$, there exists a $P \succ 0$ that solves:

$$A^T P A - P = -Q$$

Hint: Try to bound the norm $|x_n|^2$ as a function of n . You may also need the linear algebra fact that, for a symmetric matrix M :

$$\lambda_{\min}|x|^2 \leq x^T M x \leq \lambda_{\max}|x|^2$$

Here, λ_{\min} is the smallest eigenvalue of M , and λ_{\max} is the largest eigenvalue of M . (Note that we can talk about smallest and largest without complications because the eigenvalues of M are all real.)

Note: For LTI systems in discrete-time, the equivalent of the state-transition matrix e^{At} is A^k . In other words:

$$x[n] = A^n x[0] + \sum_{k=0}^{n-1} A^{n-k} B u[k]$$

Note the similarities and differences with the continuous-time equivalent.