# ECE 515/ME 540 (Control System Theory and Design) – Homework 5

**Due:** Thursday, October 17 at 2pm

**Problem 1.** Consider the following LTI model:

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

Check controllability using:

a) the controllability matrix.

b) the rows of $\overline{B} = M^{-1}B$, where $M$ is chosen such that $M^{-1}AM$ is diagonal.

c) the Hautus-Rosenbrock condition.

You may use a calculator, MATLAB, or another computational device, but be sure you would know how to do this manually if needed.

**Problem 2.** Let $A$ be an $n \times n$ matrix and $B$ be an $n \times r$ matrix, both with real entries. Suppose $(A, B)$ is controllable. Prove or disprove the following statements. (If the statement is false, then producing a counterexample will suffice.)

a) The pair $(A^2, B)$ is controllable.

b) Let $k(\cdot)$ be a known $n$-dimensional function, piecewise continuous in $t$. Consider the model:

$$\dot{x} = Ax + Bu + k(t)$$

This model is controllable, in the sense that for any initial state $x_0$ and any target final state $x_f$, there exists a control $u(\cdot)$ that directs the system from $x_0$ to $x_f$ in finite time.

c) Given that the model $\dot{x} = Ax + Bu$ has the initial condition $x(0) = x_0 \neq 0$, it is possible to find a piecewise continuous control, defined on $[0, \infty)$, such that the model is brought to rest at $t = 1$, i.e. $x(t) = 0$ for all $t \geq 1$.

d) Suppose the model starts at $x(0) = 0$; there exists a piecewise continuous control which will bring the state to $x_f \in \mathbb{R}^n$ by time $t \geq 1$ and maintain that value $x(t) = x_f$ for all $t \geq 1$.

**Problem 3.** Define the operator $\mathcal{A}$ as follows:

$$\mathcal{A}(u) = \int_{t_0}^{t_f} \phi(t_f, \tau) B(\tau) u(\tau) d\tau$$

The domain of $\mathcal{A}$ is the set of all piecewise continuous time functions on $[t_0, t_f]$. The co-domain is $\mathbb{R}^n$.

a) Compute the adjoint $\mathcal{A}^*$.

b) Compute the composition $V = \mathcal{A} \circ \mathcal{A}^*$. Note $V : \mathbb{R}^n \to \mathbb{R}^n$, so $V$ should be a $n \times n$ matrix. What is the relationship with $V$ and the controllability Grammian $W$? Adapt Theorem 5.2.2 from the reader to a statement about $V$.