ECE/CS 541
Computer System Analysis:
Introduction to Combinatorial Methods

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Fall 2018

Learning Objectives

- Or what is this course about?

- At the start of the semester, you should have
  - Basic programming skills (C++, Python, etc.)
  - Basic understanding of probability theory (ECE313 or equivalent)

- At the end of the semester, you should be able to
  - Understand different system modeling approaches
    - Combinatorial methods, state-space methods, etc.
  - Understand different model analysis methods
    - Analytic/numeric methods, simulation
  - Understand the basics of discrete event simulation
  - Design simulation experiments and analyze their results
  - Gain hands-on experience with different modeling and analysis tools

Announcements and Reminders

- HW1 is out
  - Covers the probability review
  - Prepare you for the probability quiz
  - Due on **September 18, 2018 at the start of class**

- Probability quiz on **September 20, 2018**
  - First 30 minutes of class

- Project Proposals due near the first week of October
  - Start forming groups and thinking about your projects
  - Come to office hours for discussions

Objectives for this Module

- Introduce combinatorial (non state-space) methods of modeling
- Develop and formulate models of system reliability
- Introduce different reliability formalisms
- Combinatorial models for improved testing research at Internet scale
  - Technique generated out of UC Santa Cruz and adopted by Netflix

Lecture Outline

- Assumptions for combinatorial modeling
- Review definition of reliability
- Failure rate
- System reliability
  - Maximum
  - Minimum

Introduction to Combinatorial Methods

- Combinatorial validation methods are the simplest kind of analytical/numerical techniques and can be used for reliability and availability modeling under certain assumptions.

- Assumption 1:
  - The system being studied is composed of several elementary units, called *components*.

- Assumption 2:
  - The components of the system fail in a statistically independent manner. For availability analysis, they can be repaired independently.

- When these assumptions hold, simple formulas for reliability and availability exist.

Choosing Validation Techniques cont.

Criterion	Combinatorial	State-Space-Based	Simulation	Measurement
Stage	Any	Any	Any	Post-prototype
Time	Small	Medium	Medium	Varies
Tools	Formulae, tools	Languages & tools	Languages & tools	instrumentation
Accuracy	Low	Moderate	Moderate	high
Comparisons	Easy	Moderate	Moderate	Difficult
Cost	Low	Low/medium	Medium	High
Scalability	High	Low/medium	Medium	low

Reliability

- One key to building highly available systems is the use of reliable components and systems.

- Reliability:
  - The *reliability* of a system at time t ($R(t)$) is the probability that the system operation is proper throughout the interval $[0,t]$.

- Probability theory and combinatorics can be directly applied to reliability models.

- Let X be a random variable representing the time to failure (TTF) of a component. The reliability of the component at time t is given by

$$R_X(t) = P(X > t) = 1 - P(X \le t) = 1 - F_X(t)$$

- Similarly, we can define *unreliability* at time t by

$$U_X(t) = P(X \le t) = F_X(t)$$

Failure Rate

What is the rate that a component fails at time t? This is the probability that a component that has not yet failed fails in the interval $(t, t + \Delta t)$, as $\Delta t \to 0$.

Note that we are not looking at $f_X(t)dt = P(X \in (t, t + \Delta t))$. Rather, we are seeking $P\left(X \in (t, t + \Delta t) \mid X > t\right)$

$r_X(t)$ is called the *failure rate* or *hazard rate*.

Failure Rate

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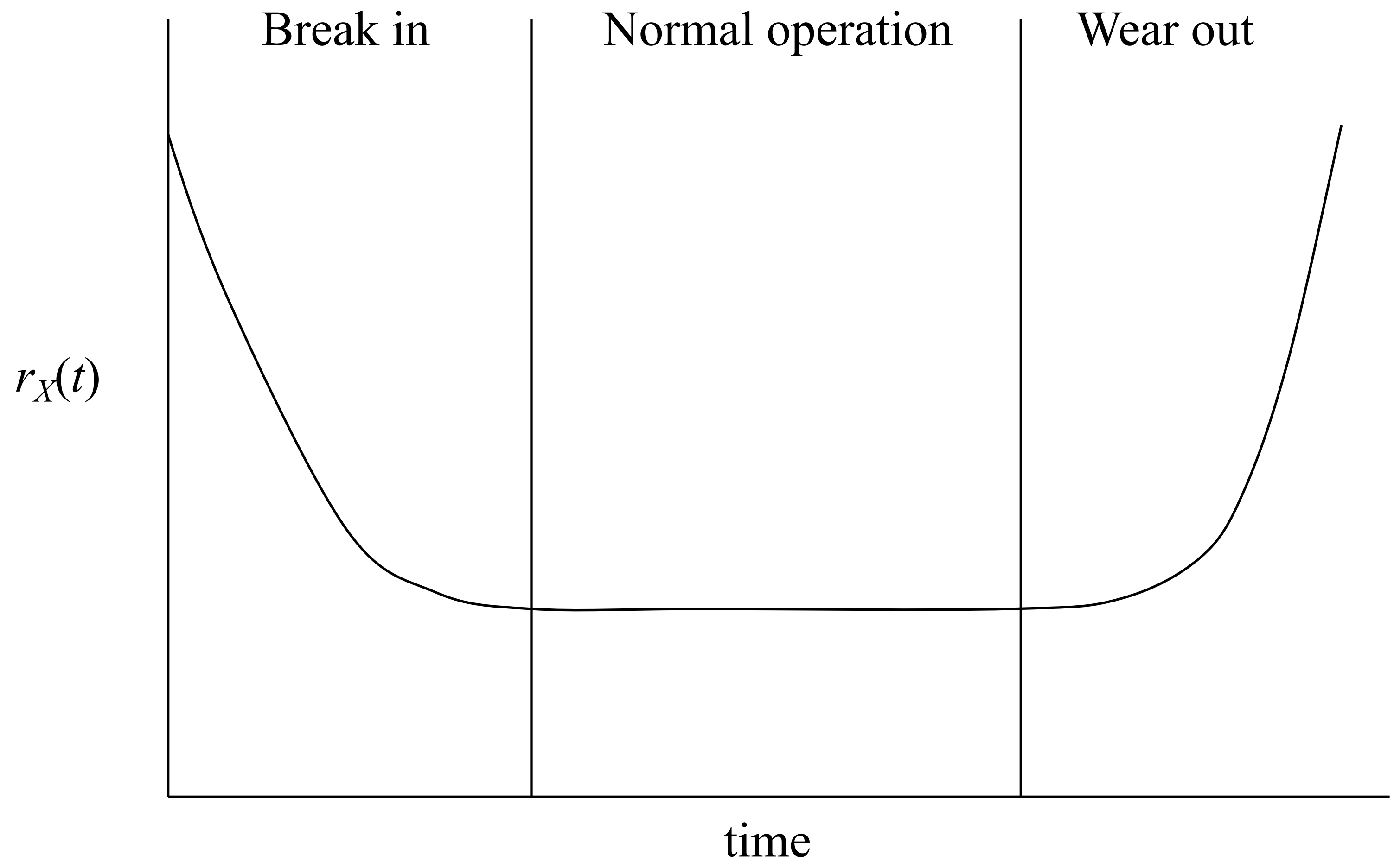
Note that we are not looking at $f_X(t)dt = P(X \in (t, t+\Delta t))$. Rather, we are seeking $P\left(X \in (t, t+\Delta t) \mid X > t\right)$

$$
\begin{aligned}
P\left(X \in (t, t+\Delta t) \mid X > t\right) &= \frac{P(X \in (t, t+\Delta t), X > t)}{P(X > t)} \\
&= \frac{P(X \in (t, t+\Delta t))}{1 - F_X(t)} \\
&= \frac{f_X(t)dt}{1 - F_X(t)} \triangleq r_x(t)dt
\end{aligned}
$$

$$
\boxed{r_x(t) = \frac{f_X(t)}{1 - F_X(t)} = \frac{f_X(t)}{R_X(t)}}
$$

$r_X(t)$ is called the *failure rate* or *hazard rate*.

Typical Failure Rate



System Reliability

While R_X can give the reliability of a component, how do you compute the reliability of a system?

System failure can occur when one, all, or some of the components fail. If one makes the *independent failure assumption*, system failure can be computed quite simply. The independent failure assumption states that all component failures of a system are independent, i.e., the failure of one component does not cause another component to be more or less likely to fail.

Given this assumption, one can determine:

1) Minimum failure time of a set of components
2) Maximum failure time of a set of components
3) Probability that k of N components have failed at a particular time t.

Maximum of *n* Independent Failure Times

Let $X_1, \ldots, X_n$ be independent component failure times. Suppose the system fails at time S if all the components fail.

Thus, $S = \max\{X_1, X_2, \ldots, X_n\}$

What is $F_s(t)$?

Maximum of *n* Independent Failure Times

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What is $F_s(t)$?

$$\begin{aligned} F_S(t) = P(S \leq t) &= P(X_1 \leq t \wedge X_2 \leq t \wedge \ldots \wedge X_n \leq t) \\ &= P(X_1 \leq t) \times P(X_2 \leq t) \times \ldots \times P(X_n \leq t) \\ &= F_{X_1}(t) F_{X_2}(t) \ldots F_{X_n}(t) \\ &= \Pi_{i=1}^{n} F_{X_i}(t) \end{aligned}$$

By definition!

By independence!

Minimum of n Independent Component Failure Times

Let $X_1, \ldots, X_n$ be independent component failure times. A system fails at time S if any of the components fail.

Thus, $S = \min\{X_1, \ldots, X_n\}$.

What is $F_S(t)$? Proof in Homework 1

$$\boxed{F_S(t) = 1 - \Pi_{i=1}^{n}(1 - F_X(t))}$$