Random Variate Generation

§1 What is a random variate?

Outcome of a random variable.

- We have seen how to produce a $U[0,1]$ R.V. but what about more useful ones (as they relate to interarrival times etc.)
- We can apply simple transformations to $U[0,1]$ to achieve that.

Example 1

Let U be a uniform $[0,1]$ R.V.

goal 1: get $X \sim$ Uniform $[a,b]$ $\longrightarrow$ $f_X(t) = \frac{1}{b-a}$

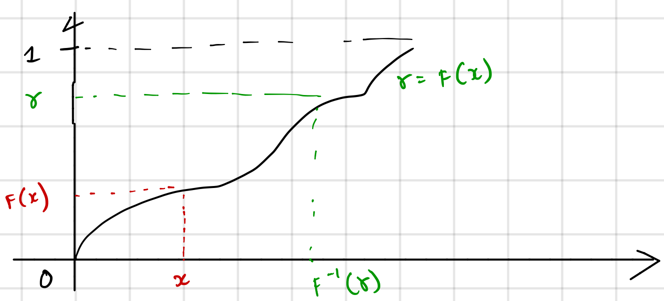
$F_X(t) = \int_a^t \frac{1}{b-a} ds = \frac{t-a}{b-a}$

let's define $X = a + (b-a)U$ Notice $U=0$ $X=a$

$U=1$ $X=b$

$$P(X < t) = P\{a + (b-a)U < t\}$$
$$= P\left\{U < \frac{t-a}{b-a}\right\}$$
$$= F_X(t)$$

- Inverse Transform Method



Let $F(x)$, $x \in \mathbb{R}$ be the CDF

$F: \mathbb{R} \longrightarrow [0,1]$ $\rightarrow$ non-negative, monotone, continuous from the right.

$F(\infty) = 1$ $F(-\infty) = 0$

Objective of random variate generation: generate R.V. X with distribution $F_X(x)$

i.e. $P(X \leq x) = F_X(x)$ $x \in \mathbb{R}$.

The inverse of F, $F^{-1}: [0,1] \longrightarrow \mathbb{R}$

$$F^{-1}(y) = \min\{x : F(x) \geq y\} \quad y \in [0,1]$$

- F^{-1} exists because : continuous & monotone
  $\Rightarrow$ for every γ $\exists$ a unique x with $F^{-1}(\gamma) = x$

- Proposition: Define $X := F^{-1}(U)$ and $U \sim$ Uniform $[0,1]$ then X has CDF F i.e. $P(X \le x) = F(x)$

$$P(X \le x) = P(F^{-1}(U) \le x)$$
$$= P(F[F^{-1}(U)] \le F(x))$$ [Due to monotonicity of function of smaller is smaller]

$$= P(U \le F(x))$$ [By defn of F^{-1}]
$$= F(x)$$ $\left[F_U(F(x)) = \frac{F(x) - 0}{1 - 0} = F(x) \right]$

eg: Generate continuous $Y \sim$ Uniform (a, b)

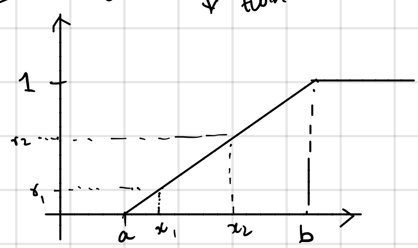
$$f_Y(x) = \frac{1}{b-a} \qquad F_Y(x) = \begin{cases} 0 & x < 0 \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

$$= F^{-1}(x) = (b-a)\,y + a$$
↳ generator function

$X \sim$ Uniform $[0,1]$
$Y \sim$ Uniform $[a,b]$

$$Y = (b-a)\,U + a$$

→ $y = mx + c$
↓ turn this 90° left



eg: Generate $Y \sim$ exponential (λ)

$$F_Y(y) = 1 - e^{-\lambda y}$$

$$\gamma = F_Y(y) = 1 - e^{-\lambda y}$$

$$1 - \gamma = e^{-\lambda y}$$
$$\ln(1-\gamma) = -\lambda y$$ → $[0,1]$ → ln of $[0,1]$ is -ve.
$$y = \frac{-1}{\lambda} \ln(1-\gamma)$$

Aside

If we have 2 independent exponentials $X_1 \sim exp(\lambda_1)$ $X_2 \sim exp(\lambda_2)$
if $\lambda_1 << \lambda_2$:

still possible to get sample from $exp(\lambda_1) > exp(\lambda_2)$

But If you want to compare a system with slow exponential rate with one that is faster :

System 1

$$X_{1,i} = \frac{1}{\lambda_1} \ln(1-u_i)$$

System 2

$$X_{2,i} = \frac{1}{\lambda_2} \ln(1-u_i)$$

Coupled using the same U_i

$X_{1,i} > X_{2i}$

eg: Weibull

$$F(x) = \begin{cases} 1 - e^{-(x/\lambda)^k} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

setup

$$u = 1 - e^{-(x/\lambda)^k}$$

$$1-u = e^{-(x/\lambda)^k}$$

$$\ln(1-u) = -\left(\frac{x}{\lambda}\right)^k$$

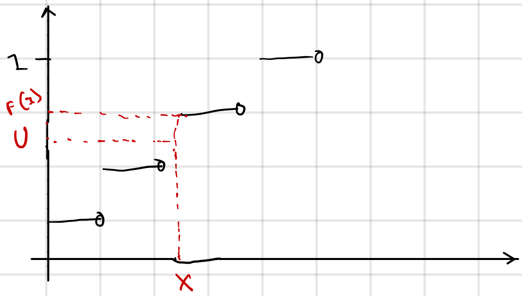
$$-\ln(1-u) = \left(\frac{x}{\lambda}\right)^k$$

$$-\lambda^k \ln(1-u) = x^k$$

$$x = -\lambda\left[\ln(1-u)\right]^{1/k}$$

Discrete transformations

$F^{-1}(y) = \min\{x : F(x) \geq y\}$



$U = F(x)$

$F^{-1}(U) = x = \min\{x : F(x) \geq U\}$

↳ round up.

eg: $Y \sim$ geometric (p)

$$F_Y(k) = 1 - (1-p)^{k+1}$$

$$F^{-1}(u) = \min_k \{1-(1-p)^{k+1} \geq u\}$$

$$= \min_k \{(1-p)^{k+1} \leq 1-u\}$$

$$\{\log(1-u) \geq (k+1)\log(1-p)\}$$

$$\left\{\frac{\log(1-u)}{\log(1-p)} \geq k+1\right\}$$

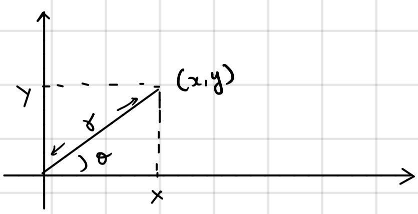
$$\left\{\left\lceil \frac{\log(1-u)}{\log(1-p)} \right\rceil - 1 = k\right\}$$

- Normal $(0, 1)$ - Standard Normal

$$F(x) = \int_{-\infty}^{x} \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

→ Hard to do because inverting CDF is hard

Polar coordinates



$r^2 = x^2 + y^2$

$\tan(\theta) = y/x$

$x = r\cos\theta$

$y = r\sin\theta$

Box - Muller Method

Consider two Independent $N(0,1)$

$X \sim N(0,1)$

$Y \sim N(0,1)$

What is their joint density function.

$$f(x,y) = f(x) \cdot f(y) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \quad \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

$$= \frac{1}{2\pi} \quad e^{-\frac{1}{2}(x^2+y^2)}$$

$$= \frac{1}{2\pi} \quad e^{-r^2/2} \qquad (r^2 = x^2+y^2)$$

This is still a product of two pdfs but can we think of them as different pdfs than normals, something that we know

$\frac{1}{2\pi}$ is pdf of $U(0, 2\pi)$

What about $e^{-r^2/2}$

Let's look at the CDF of an exponential R.V. Z

$$P(Z \le r^2) = 1 - e^{-r^2/2}$$

$$d/dr \left(1 - e^{-r^2/2}\right) = 2 \cdot \frac{1}{2} \cdot e^{-r^2/2}$$
$$= e^{-r^2/2}$$

$R^2 \sim \exp(1/2)$ with pdf $e^{-r^2/2}$

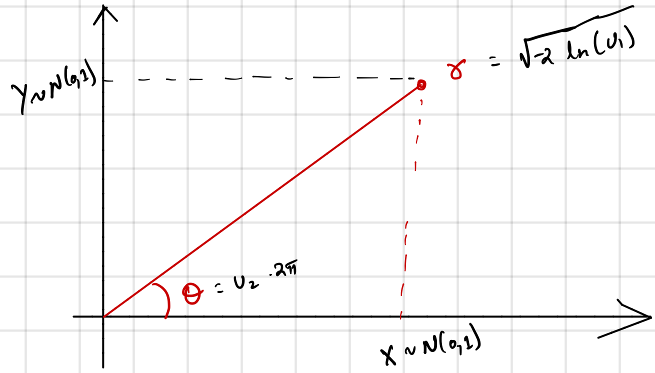
Remember, we can sample an exponential from uniform by $-\frac{1}{\lambda} \ln(1-u) \sim -\frac{1}{\lambda} \ln(u)$

So sampling for $R^2 = -2 \ln(U_1)$

$$R = \sqrt{-2 \ln(U_1)}$$

Also remember, for Uniform (a,b) $(b-a)U + a$

$(0, 2\pi)$ $2\pi \cdot U_2 - \theta$



Now polar $\longrightarrow$ rectangular coordinates

$$x = r\cos\theta = (-2\ln U_1)^{1/2} \cdot \cos(U_2 \cdot 2\pi)$$
$$y = r\sin\theta = (-2\ln U_1)^{1/2} \sin(U_2 \cdot 2\pi)$$

$\therefore$ 2 independent Uniform R.V $\longrightarrow$ 2 independent $N(0,1)$

- Transform $Z \sim N(0,1) \rightarrow X \sim N(\mu, \sigma)$

$$X = \sigma Z + \mu$$

Acceptance - Rejection Method / Rejection Sampling

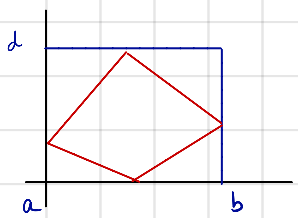
Basic form

a) Sample x from "easy" distribution
b) accept x if it meets some criteria | reject otherwise

Simplest form: We want $X \sim U[1/4, 1]$

Step 1: Generate $R_i \sim U[0,1]$
Step 2a: if $R_i \geq 1/4$ accept
Step 2b: else: reject & return to step 1
Step 3: Go to Step 1 or Stop

example:

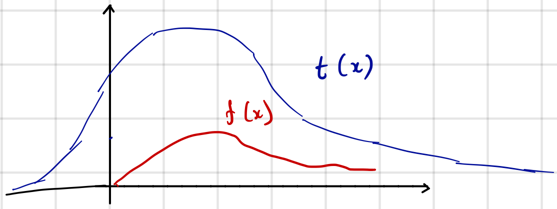


1. Sample (x, y) a point
$x \sim U[a,b]$
$y \sim U[a,d]$

2. Use geometry to determine whether (x,y) falls within red polygon $\rightarrow$ accept / reject.

Generalization

Goal: generate X with density function f



Let t, f be density functions
- t is easy to sample
- $f(x)$ can be computed

$t(x)$ must majorize $f(x)$ i.e. $t(x) \geq f(x) \ \forall x$

then $t(x) \geq 0$ but $\int_{-\infty}^{\infty} t(x)\, dx \geq f(x)\, dx = 1$ so $t(x)$ isn't a density function

Set $c = \int_{-\infty}^{\infty} t(x)\, dx \geq 1$ → area under curve

Define $r(x) = t(x)/c \ \ \forall x$
↓
is a density since it integrates to 1

Algorithm

1. generate sample Y having density r
2. generate sample $U \sim U(0,1)$
3. If $U \leq f(y)/t(y)$ return $x = y$ (what we want) and stop

   else reject & return to step 1

Note: Since t majorizes f

$$f(y)/t(y) \leq 1 \rightarrow \left[0, \frac{f(y)}{t(y)}\right]$$

$U \leq$ ↓ may or may not occur

$$P(\text{acceptance}) = \int_{-\infty}^{\infty} r(x) \cdot \frac{f(x)}{t(x)}\, dx$$

$$= \frac{1}{c} \cdot 1 = 1/c$$

So want C as close to 1. Best fit.

Convolution Method

- Some random V. are naturally expressed as sums of others
  - Sum of N - bernoulli rvs is binomial → # of successes
    just do N-trials & sum outcomes
  - Sum of k - exponentials is erlang - k

$X \sim erlang(k, \theta) \rightarrow$ Sum of k - independent exponential R.V. $X_i (i = 1, k)$
$E[X_i] = 1/k\theta$

$$X = \sum_{i=1}^{k} X_i$$

We already saw using inverse transform how to generate exponentials

$$X = -1/\lambda \ln(1-u)$$

$\lambda = k\theta$

$$X = \sum_{i=1}^{k} -1/k\theta \ln(u_i)$$

$$= -1/k\theta \sum_{i=1}^{k} \ln(u_i)$$

$$= -1/k\theta \ln\left(\prod_{i=1}^{k} U_i\right)$$