University of Illinois Fall 2018

ECE 586GT: Exam II

Monday, December 10, 2018
7:00 p.m. — 8:30 p.m.
2013 Electrical Engineering Building

1. **[40 points]** Recall the standard model for infinite repeated play of the prisoners' dilemma game:

	C	D
C	1,1	-1,2
D	2,-1	0,0

with payoff of player i given by $J_i = (1-\delta)\sum_{t=1}^{\infty} g_i(s_t)\delta^{t-1}$ for some $\delta \in (0,1)$. Let $s^T = (s_1^T, s_2^T)$ denote the profile of trigger strategies, such that $s_i^T(h_t) = C$ if $h_t = ((C,C),\ldots,(C,C))$ and $s_i^T(h_t) = D$ otherwise. Consider the following variation, for some $\epsilon \in (0,1)$. If a player attempts to play action C in a stage t, due to a communication error, the player's publicly announced action is D with probability ϵ. If a player attempts to play action D no error is possible, so the publicly announced action for the player is also D. If both players play C the errors happen independently for the two players. The payoffs of the players are based on the publicly announced actions. When s^T is used, it is applied to the history of publicly announced actions (not on the actions attempted by the players). The goal of this problem is to find conditions on ϵ and δ such that s^T is subgame perfect.

(a) Suppose both players follow s^T with one exception: namely, player 1 deviates in the first stage. What is the expected payoff of player 1? (Your answer should depend on ϵ and δ.)

Solution: For any stage after stage 1, both players play D, so the stage payoff is zero. For stage 1, player 1 gets payoff 2 with probability $1-\epsilon$ and 0 with probability ϵ, so, including the weight factor, the expected payoff is $2(1-\epsilon)(1-\delta)$.

(b) Suppose both players follow s^T. Let X denote the first stage such that at least one player is publicly announced to have played D. What are $P\{X > t\}$, $P\{X = t\}$, and $\alpha \triangleq E[g_i(s_t^T)|X = t]$ for $t \geq 1$?

Solution: $P\{X > t\} = (1-\epsilon)^{2t}$, $P\{X = t\} = (2\epsilon - \epsilon^2)(1-\epsilon)^{2(t-1)}$. Given $X = t$, the publicly announced plays are $(D,C), (C,D)$, or (D,D), with probabilities proportional to $\epsilon(1-\epsilon), \epsilon(1-\epsilon)$, or ϵ^2, respectively, giving reward 2, -1, or 0 to player 1, respectively. Hence, $\alpha = E[g_i(s_t^T)|X = t] = \frac{2\epsilon(1-\epsilon)-\epsilon(1-\epsilon)+0\epsilon^2}{2\epsilon-\epsilon^2} = \frac{1-\epsilon}{2-\epsilon}$.

(c) For what values of δ and ϵ is s^T a subgame perfect equilibrium? (Due to time restrictions, you can leave your answer in terms of infinite sums.)

Solution: By the one-step deviation principle, s^T is subgame perfect if and only if it satisfies the one step deviation principle. Deviating for any history such that a D occurred would not help a player, because the other player will be playing D forever more. To check deviations for histories with no D's, it suffices to check deviation at time 1 (for time t just multiply expected subgame payoffs by δ^{t-1}). Thus, s^T is subgame perfect if the expected payoff of player 1 under s^T is greater than or equal to the expected payoff found in (a). Based on part (c), the payoff of player 1 under s^T is:

$$\begin{aligned}
&(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\left(P\{X>t\}+\alpha P\{X=t\}\right)\\
&=(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\left((1-\epsilon)^{2t}+(1-\epsilon)^{2(t-1)}(2\epsilon-\epsilon^2)\alpha\right)\\
&=(1-\delta)\left((1-\epsilon)^2+(2\epsilon-\epsilon^2)\alpha\right)\sum_{t=1}^{\infty}[\delta(1-\epsilon)^2]^{t-1}\\
&=(1-\delta)\left[\frac{(1-\epsilon)^2+(2\epsilon-\epsilon^2)\alpha}{1-\delta(1-\epsilon)^2}\right]\\
&=\frac{(1-\delta)(1-\epsilon)}{1-\delta(1-\epsilon)^2}
\end{aligned}$$

Hence s^T is subgame perfect if and only if $\frac{1}{1-\delta(1-\epsilon^2)}\geq 2$, or, equivalently, $\delta\geq\frac{1}{2(1-\epsilon)^2}$.

(d) What is $\lim_{\delta\to 1} J_i(s^T)$ for either player i and a fixed $\epsilon\in(0,1)$? Explain. (Hint: This can be answered independently of parts (a) - (c).)
Solution: After a finite number of stages both players will play D and get reward zero per stage, so $\lim_{\delta\to 1} J_i(s^T)=0$ for both players. (In other words, the trigger strategy profile s^T in the long run does no better than both players playing D all the time. However, a strictly positive limit could be obtained for ϵ and $1-\delta$ sufficiently small as follows. Let L be a positive integer. After a D appears, both players play D for L subsequent stages, and then the trigger strategy resets. This problem gives an inkling of a rich literature on realization theory with noisy observed actions.)

2. **[30 points]** Consider the sale of k identical objects to n bidders such that $1\leq k\leq n-1$. Suppose each bidder can be allocated at most one object, and bidder i has value v_i for any one of the objects.

(a) Describe the VCG mechanism for selling the objects. In other words, describe the VCG allocation rule and associated payment rule. Explain how your solution follows from the general form of VCG mechanisms, and simplify it as much as possible.
Solution: Bid i, $\hat{v}_i$, is in $S_i=\mathbb{R}$. $\mathcal{C}$ is the set of subsets of $[n]$ of cardinality k, (i.e. $\mathcal{C}=\binom{[n]}{k}$.) Given a bid vector $\hat{v}=(\hat{v}_i)_{i\in I}$, $g(\hat{v})=c^*=\arg\max_{c\in\mathcal{C}}\sum_{j\in c}\hat{v}_j$. More explicitly, $c^*=\{\pi^*(1),\ldots,\pi^*(k)\}$ for some permutation $\pi^*:[n]\to[n]$ such that $\hat{v}_{\pi^*(1)}\geq\ldots\geq\hat{v}_{\pi^*(n)}$. The payment of buyer i can be taken to be the decrease in welfare of the other bidders due to the presence of bidder i. If $i\notin c$ then $m_i=0$: bidder i has no impact on the other bidders. If $i\in c*$, with bidder i present, $k-1$ other bidders are allocated objects, and if bidder i were not present, $k-1$ objects would be allocated to bidders with the same values as before, and a k^{th} object would be allocated to a bidder with bid $\hat{v}(\pi^*(k+1))$. Thus, $m_i=\hat{v}(\pi^*(k+1))$ for all $i\in c^*$. In words, the items are sold to the k highest bidders and the price to each of them is the $k+1^{st}$ highest price.

(b) Describe the revenue optimal, incentive compatible, individually rational selling mechanism for selling the objects, under the assumption that the valuations are independent and uniformly distributed over the interval $[0,1]$. To be definite, suppose that the seller must sell all k objects, or equivalently, the value to seller of any object is $-\infty$. Explain how your solution follows from the general form of revenue optimal mechanisms, and simplify it as much as possible.
Solution: The virtual valuation for a bid $x\in[0,1]$ is given by $\psi(x)=x-\frac{1-F(x)}{f(x)}=2x-1$. The function ψ is increasing and hence regular. The optimal auction is to select $c\in\mathcal{C}$

to maximize $\sum_{i\in c}\psi(\hat{v}_i)$, and the payment m_i of a bidder allocated an object is the min to win bid for that bidder. Since the functions ψ_i are strictly increasing, the auction reduces to the VCG auction found in part (a), for both the allocation and payment rules.

3. **[30 points]** The cooperative games below are for a set of four players, $I = \{1,2,3,4\}$. As usual, assume transferrable utilities. Simplify your answers as much as possible.

   (a) (10 points) Suppose $v_1 = (v_1(S))_{S\subset I}$ is given by

$$v_1(S) = \begin{cases} 1 & \text{if } \{1,2\}\subset S \text{ or } \{3,4\}\subset S \\ 0 & \text{else} \end{cases}.$$

   Is (I, v_1) cohesive? Find the core of (I, v_1).
   Solution: The game is not cohesive because $v_1\{1,2\} + v_1\{3,4\} = 2 > v_1(I)$. Consequently the core is empty.

   (b) (20 points) Suppose $v = v_1 + v_2$, where v_1 is defined in (a), and

$$v_2(S) = \begin{cases} 1 & \text{if } \{1,3\}\subset S \text{ or } \{2,4\}\subset S \\ 0 & \text{else} \end{cases}.$$

   For example, the players could be working for two days, with values v_1 the first day and v_2 the second day. Is (I, v) cohesive? Find the core of (I, v).
   Solution: To test whether the game is cohesive, by symmetry, it suffices to consider a few partitions of I. Note that $v(I) = 2$.

$$\begin{aligned} v\{1,2\} + v\{3,4\} &= 1 + 1 \le 2 \\ v\{1,4\} + v\{2,3\} &= 0 \le 2 \\ v\{1,2,3\} + v\{4\} &= 2 + 0 \le 2 \end{aligned}$$

   Hence, the game is cohesive.
   The core constraints are as follows:

$$\begin{aligned} x_i &\ge 0 \text{ for } 1 \le i \le 4 \\ x_1 + x_2 &\ge 1 \\ x_3 + x_4 &\ge 1 \\ x_1 + x_3 &\ge 1 \\ x_2 + x_4 &\ge 1 \\ x_1 + x_2 + x_3 &\ge 2 \\ x_1 + x_2 + x_4 &\ge 2 \\ x_1 + x_3 + x_4 &\ge 2 \\ x_2 + x_3 + x_4 &\ge 2 \\ x_1 + x_2 + x_3 + x_4 &= 2 \end{aligned}$$

   Adding the four constraints involving three x's yields $x_1 + x_2 + x_3 + x_4 \ge 8/3$, contradicting the equality constraint. Thus, the core is the empty set.