

ECE 586BH: Problem Set 4
 Extensive form (aka sequential) games

Due: Tuesday, October 30, at beginning of class

Reading: Course notes, Sections 4.1 and 4.2

1. **[Variations of pirate game]**

Five pirates, A, B, C, D, E, in a boat at sea must decide how to allocate 100 gold coins among themselves. The pirates are lettered in order of seniority, with A being the most senior. The pirate protocol is that the most senior pirate proposes an allocation of coins to all pirates, with a nonnegative integer number of coins going to each pirate. Then all pirates, including the proposer, vote on the proposal. If a majority (meaning strictly more than half the number voting) approve the proposal, then the proposal is implemented and the game ends. Otherwise, the proposer is thrown overboard and votes no more, and the remaining pirates use the same protocol again, with the most senior remaining pirate being the next proposer, and so on. Pirates base decisions on four factors. First, each pirate prefers to not be thrown overboard, even if the pirate receives no coins. Second, given a pirate is not thrown overboard, the pirate prefers to have more coins. Third, if in any round the pirate would do as well to vote either way, the pirate will vote no in order to increase the chance the proposer is thrown overboard. Finally, pirates follow the protocol, but otherwise don't trust each other and so can't enter into binding side agreements with each other. (This differs from the version on wikipedia https://en.wikipedia.org/wiki/Pirate_game because we assume that in case of a tie vote the proposer is thrown overboard.)

- (a) What is the payoff vector for a subgame perfect strategy? Is such payoff vector unique?
- (b) Repeat part (a), but now assume that a proposal is accepted if and only if the number of pirates voting to accept the proposal is $2/3$ or more of the number of pirates voting (including the proposer).

2. **[Market power in territory control game, continued]**

Recall the territory control game for a graph $G = (V, E)$ and $n \geq 2$ players, defined in problem set 1 and revisited in problem set 2. As in problem set 2, suppose that players 2 and 3 are grouped together into a single player that selects two vertices. For this problem, we refer to the combination of players 2 and 3 as a single player, namely player 2. The new aspect for this problem is the order that the selections are made, making it an extensive form game. First, player 2 declares a vertex at which the player will place a restaurant. Then player 1 declares a vertex for placement of a restaurant. Then player 2 declares a vertex for placement of the other restaurant of player 2. Consider the special case of a line graph $V = \{1, 2, \dots, m\}$. Keep in mind that one vertex can be selected multiple times.

- (a) Draw the game tree for the special case $m = 2$ and describe the set of all subgame perfect equilibria in pure strategies (i.e. those strategies that can be produced by backwards induction (aka dynamic programming)).
- (b) Let $F_{1,m}^*$ denote the value of the game for player 1, which is well defined because the game has two players and the sum of payoffs is constant. Identify $\lim_{m \rightarrow \infty} F_{1,m}^*/m$,

which is the fraction of customers attracted to the restaurant of player 1. (Hint: The game can be solved numerically on a computer for small m , but you can also think of the case m is large and explore backwards from the end of the game. To set notation, let v_2 be the first choice for player 2, let v_1 be the choice of player 1, and let v_3 be the second choice for player 2. Assume without loss of generality that $v_2 \geq (n + 1)/2$, in which case it is easy to see that a best response v_1 for player 1 will satisfy $v_1 \leq v_2$. Then suppose $v_1 \approx xm$ and $v_2 \approx ym$ with $0 \leq x \leq y$ and $y \geq 1/2$. Don't forget the possibility $v_1 = v_2$.)

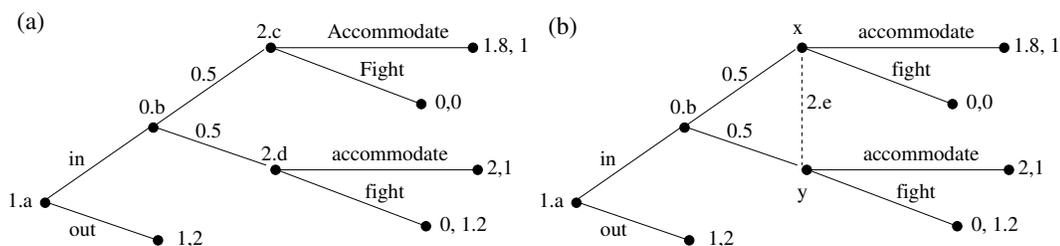
3. [Call my bluff card game]

Consider the call my bluff card game in the class notes (Examples 4.5 and 4.8).

- (a) Find the expected payoff for player 1 at Nash equilibrium. Also, in order to get some practice with definitions, identify the behavioral strategy profile σ and the belief vector μ such that the assessment (σ, μ) is the sequential equilibrium.
- (b) Consider the following variation of the game. Suppose player 2 is given a third possible action in case player 1 raises. Namely, if player 1 raises, player 2 can (re)double the stakes. In other words, if player 1 raises (proposal to double the stakes) and player 2 doubles, then the payoff vector is $(4,-4)$ if the card is red and $(-4,4)$ if the card is black. Suppose if player 2 doubles, player 1 must accept it and the game ends. As before, if player 1 checks then player 2's action is irrelevant. Give both the extensive and normal form version of the game and determine the expected value of the game to player one.

4. [Entry deterrence game with random cost]

Consider the following two variations of the entry deterrence game discussed in the notes. The node with label 0.b is controlled by nature which selects each possible outcome with probability 0.5. For version (a) player 2 knows the random outcome of nature when it needs to select an action, and for version (b) player 2 doesn't know it.



- (a) Find the subgame perfect equilibria and expected payoff vector for version (a).
- (b) Find the sequential equilibrium and associated expected payoff vector for version (b). In order to get some practice with definitions, identify the behavioral strategy profile σ and the belief vector μ such that the assessment (σ, μ) is the sequential equilibrium.

5. [Trembling hand perfect strategies and perfect information games]

- (a) Consider a perfect information sequential game such that any player controls at most one node along any instance of the game (i.e. along any path from root to a leaf of the game). Show that a trembling hand perfect behavioral strategy profile σ is subgame perfect. You may start from scratch or use a result from the notes.
- (b) Give an example of a game satisfying the conditions of the first sentence in (a) showing the converse to (a) is not true: there is a subgame perfect equilibrium that is not THP.