ECE588 - Electricity Resource Planning
Homework 2 Markov Models

1. **Problem 1** :
   We know that,
$$\mathcal{F}_i = \sum_{j \neq i} \mathcal{F}_{ij} = \sum_{j \neq i} \lambda_{ij} p_j \tag{1}$$
   In long term we also know that,
$$\sum_{j \neq i} \lambda_{ij} p_j = \lambda_i p_i \tag{2}$$
   In a homogeneous Markov chain holds that,
$$\lambda_i = \sum_{j \neq i} \lambda_{ji} \tag{3}$$
   Then, from (2) and (3) we obtain,
$$\mathcal{F}_i = \lambda_i p_i = p_i (\sum_{j \neq i} \lambda_{ji}) \tag{4}$$

2. **Problem 2** :
   - (a)

     We got from equilibrium equation in long time,
$$\begin{bmatrix} (-\lambda_{21} - \lambda_{31}) & \mu_{12} & \mu_{13} \\ \lambda_{21} & (-\mu_{12} - \lambda_{32}) & \mu_{23} \\ \lambda_{31} & \lambda_{32} & (-\lambda_{13} - \lambda_{23}) \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$
     And the condition over probabilities,
$$P_1 + P_2 + P_3 = 1 \tag{6}$$
     Using two equation of system (5) and (6), we obtain for P_1 and P_2,
$$\begin{bmatrix} (\lambda_{21} + \lambda_{31} + \mu_{13}) & (\mu_{13} - \mu_{12}) \\ (\mu_{23} - \lambda_{21}) & (\mu_{12} + \mu_{23} + \lambda_{32}) \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \mu_{13} \\ \mu_{23} \end{bmatrix} \tag{7}$$
     Then using inverse of 2×2 matrix, we obtain for P_1 and P_2,
$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{Det} \begin{bmatrix} (\mu_{12} + \mu_{23} + \lambda_{32}) & \mu_{12} - \mu_{13} \\ \lambda_{21} - \mu_{23} & (\lambda_{21} + \lambda_{31} + \mu_{13}) \end{bmatrix} \begin{bmatrix} \mu_{13} \\ \mu_{23} \end{bmatrix} \tag{8}$$
     Where $Det = (\mu_{12} + \mu_{23} + \lambda_{32})(\lambda_{21} + \lambda_{31} + \mu_{13}) - (\mu_{12} - \mu_{13})(\lambda_{21} - \mu_{23})$, With all this, we obtain for P_1, P_2 and P_3 ,
$$P_1 = ((\mu_{12} + \lambda_{32})\mu_{13} + \mu_{12}\mu_{23}) / Det \tag{9}$$
$$P_2 = ((\lambda_{21} + \lambda_{31})\mu_{23} + \lambda_{21}\lambda_{13}) / Det \tag{10}$$
$$P_3 = ((\lambda_{31} + \lambda_{21})\lambda_{32} + (\lambda_{31} - \mu_{31})\mu_{12} + \mu_{13}\mu_{23}) / Det \tag{11}$$

Looking the space transition model, and using the definition of $\mathcal{D}_i$, we obtain that,

$$\mathcal{D}_1 = \frac{1}{\lambda_{21} + \lambda_{31}} \tag{12}$$
$$\mathcal{D}_2 = \frac{1}{\lambda_{32} + \mu_{12}} \tag{13}$$
$$\mathcal{D}_3 = \frac{1}{\mu_{13} + \mu_{23}} \tag{14}$$

Finally the frequencies are,

$$\mathcal{F}_1 = (\lambda_{21} + \lambda_{31})P_1 \tag{15}$$
$$\mathcal{F}_2 = (\lambda_{32} + \mu_{12})P_2 \tag{16}$$
$$\mathcal{F}_3 = (\mu_{13} + \mu_{23})P_3 \tag{17}$$

- (b)

If we perform a combining state between 2 and 3 we need to assume the following rules:

- The Probability to be in 2 and 3 has to be equal to the new combined state $\tilde{2}$.
- The frequency to go from 1 to $\tilde{2}$ has to be equal to the frequency to go from 1 to 2 and 3.
- The frequency to go from $\tilde{2}$ to 1 has to be equal to the frequency to go from 2 to 1 plus 3 to 1

Using the assumptions we obtain that in combined states C,

$$\lambda_{Ci} = \sum_{j \in C} P_j \lambda_{ij} \tag{18}$$
$$\lambda_{iC} = \frac{\sum_{j \in C} P_j \lambda_{ij}}{\sum_{j \in C} P_j} \tag{19}$$

So we the new transition rates will be in combined states,

$$\lambda = \lambda_{21} + \lambda_{31} \tag{20}$$

and

$$\mu = \frac{P_2\lambda_{12} + P_3\lambda_{13}}{P_2 + P_3} \tag{21}$$

The probabilities will be,

$$\tilde{P}_1 = P_1 = \frac{\mu}{\mu + \lambda} \tag{22}$$
$$\tilde{P}_2 = P_2 + P_3 = \frac{\lambda}{\mu + \lambda} \tag{23}$$

The frequencies,

$$\tilde{\mathcal{F}}_1 = P_1(\lambda_{21} + \lambda_{31}) = P_1\lambda \tag{24}$$
$$\tilde{\mathcal{F}}_2 = (P_2\lambda_{12} + P_3\lambda_{13}) = \tilde{P}_2\mu \tag{25}$$

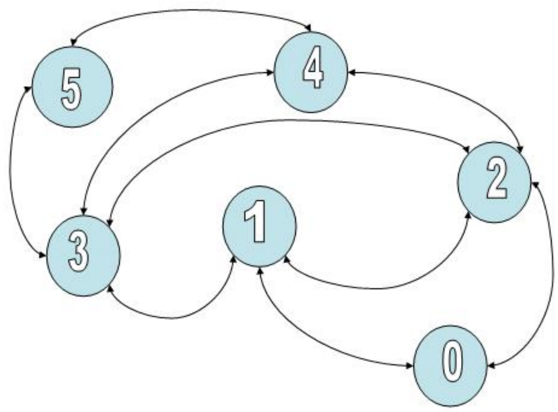
and finally $\mathcal{D}_i$,

$$\tilde{\mathcal{D}}_1 = \frac{1}{\lambda} \tag{26}$$
$$\tilde{\mathcal{D}}_2 = \frac{1}{\mu} \tag{27}$$

3. **Problem 3** :

- (a)

  In this case we can use like state variable the number of unit outs, the transition diagram will be,



- (b)

  The transition rates will be,

$from/to$	0	1	2	3	4	5
0		5λ	10λ			
1	μ		4λ	6λ		
2	μ	2μ		3λ	3λ	
3		3μ	3μ		2λ	λ
4			6μ	4μ		λ
5				10μ	5μ	

  Where we supposed that repair or fails two unit have the same rate that one unit, like a big unit with twice of capacities but the same fails and repair rates. And the logic between the transition rates is the following, to go from 0 to 2 we can 'build' this big unit in $5!/(3!2!) = 10$ different ways taking the available 5 units, for that reason the transition is 10λ. To go from 1 to 3 we can 'build' the big unit in $4!/(2!2!) = 6$ different ways taking the 4 available units, so the transition is 6λ. The other transition are done in a similar logic.

  The probabilites can be calculated roughly speaking using binomial distribution, this is valid I think only like an approximation of the process because binomial distribution gives the exact answer when we allowed only transition of 1 unit off or up:

$$P_0 = 0.96^5 = 0.81537 \tag{28}$$
$$P_1 = 5(1 - 0.96)0.96^4 = 0.1698 \tag{29}$$
$$P_2 = 10(1 - 0.96)^2 0.96^3 = 0.01415 \tag{30}$$
$$P_3 = 10(1 - 0.96)^3 0.96^2 = 0.000589 \tag{31}$$
$$P_4 = 5(1 - 0.96)^4 0.96 = 0.00001228 \tag{32}$$
$$P_5 = (1 - 0.96)^5 = 0.000000102 \tag{33}$$

An alternate solution to find P_i in a careful way is write the transition system $\Lambda\vec{P} = 0$ and find the probabilities using the condition $\sum P_i = 1$.

$$\Lambda\vec{P} = \begin{bmatrix} (-15\lambda) & \mu & \mu & 0 & 0 & 0 \\ 5\lambda & (-10\lambda-\mu) & 2\mu & 3\mu & 0 & 0 \\ 10\lambda & 4\lambda & (-3\mu-6\lambda) & 3\mu & 6\mu & 0 \\ 0 & 6\lambda & 3\lambda & (-6\mu-3\lambda) & 4\mu & 10\mu \\ 0 & 0 & 3\lambda & 2\lambda & (-\lambda-10\mu) & 5\mu \\ 0 & 0 & 0 & \lambda & \lambda & (-15\mu) \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \vec{0} \tag{34}$$

To find the solution to this system, we need to change one of the equation because the matrix is singular by the equation $\sum_{i=0}^{5} P_i = 1$. Replacing the values of μ and λ the system to solve is,

$$\begin{bmatrix} -6 & 9.6 & 9.6 & 0 & 0 & 0 \\ 2 & -13.6 & 19.2 & 28.8 & 0 & 0 \\ 4 & 1.6 & -31.2 & 28.8 & 57.6 & 0 \\ 0 & 2.4 & 1.2 & -58.8 & 38.4 & 96 \\ 0 & 0 & 1.2 & 0.8 & -96.4 & 48 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{35}$$

$$P_0 = 0.605652 \tag{36}$$
$$P_1 = 0.271061 \tag{37}$$
$$P_2 = 0.107472 \tag{38}$$
$$P_3 = 0.014294 \tag{39}$$
$$P_4 = 0.001478 \tag{40}$$
$$P_5 = 0.000044 \tag{41}$$

We can see that there is a change from using binomial distribution, an that makes sense because, for example, if now is allowed fail 1 or 2 units the probability in state with 0 units outs has to decreased.
From the transition table, we can evaluate $\mathcal{D}_i$

$$\mathcal{D}_0 = \frac{1}{15\lambda} \tag{42}$$
$$\mathcal{D}_1 = \frac{1}{10\lambda+\mu} \tag{43}$$
$$\mathcal{D}_2 = \frac{1}{6\lambda+3\mu} \tag{44}$$
$$\mathcal{D}_3 = \frac{1}{6\mu+3\lambda} \tag{45}$$
$$\mathcal{D}_4 = \frac{1}{10\mu+\lambda} \tag{46}$$
$$\mathcal{D}_5 = \frac{1}{15\mu} \tag{47}$$

To evaluate the frequencies we use the relation $\mathcal{F}_i\mathcal{D}_i = P_i$ then,

$$\mathcal{F}_0 = (15\lambda)P_0 \tag{48}$$
$$\mathcal{F}_1 = (10\lambda+\mu)P_1 \tag{49}$$
$$\mathcal{F}_2 = (6\lambda+3\mu)P_2 \tag{50}$$
$$\mathcal{F}_3 = (6\mu+3\lambda)P_3 \tag{51}$$
$$\mathcal{F}_4 = (10\mu+\lambda)P_4 \tag{52}$$
$$\mathcal{F}_5 = (15\mu)P_5 \tag{53}$$

- (c)

The cumulative distribution $P\left(\sum A \leq x\right)$ is builded in the following way, the probability to have less or equal than 0 capacity will be when all units are off thats mean P_5, the probability to have less or equal to $50kW$ will be equal to all units off or 4 units off, $P_4 + P_5$, and go on. So the cumulative will be,

$$F(x) = u(x)P_5 + u(x-50)P_4 + u(x-100)P_3 + u(x-150)P_2 + u(x-200)P_1 + u(x-250)P_0 \quad (54)$$

4. **Problem 4** :

To find the transition rates we use the fact that,

$$\mathcal{D}_i = e\mathcal{D}_0 \quad (55)$$

$$\mathcal{D}_{base} = (1-e)\mathcal{D}_0 \quad (56)$$

Using the relation $\mathcal{D}_i\mathcal{F}_i = P_i$, it is possible to find that,

$$e\mathcal{D}_0(\lambda_{l_i-})P_i = P_i \quad (57)$$

Where we used the frequency of state i, then

$$\lambda_{l_i-} = \frac{1}{e\mathcal{D}_0} \quad (58)$$

In a similar way, for the base state we can evaluate first the frequency,

$$\mathcal{F}_{base} = P_{l_0}\sum_{i=1}^{L}\alpha_i\lambda_{l_0+} = P_{l_0}\lambda_{l_0+} \quad (59)$$

So in relation $\mathcal{D}_{base}\mathcal{F}_{base} = P_{l_0}$ we obtain,

$$(1-e)\mathcal{D}_0\lambda_{l_0+}P_{l_0+} = P_{l_0+} \quad (60)$$

Then,

$$\lambda_{l_0+} = \frac{1}{(1-e)\mathcal{D}_0} \quad (61)$$

To evaluate the probabilities we can reduce the system into a two states model, collapsing all states L_i into one. To do this we need to evaluate the new rates of transition, from problem (2) we find that,

$$\tilde{\lambda}_{l_0+} = \sum_{i=1}^{L}a_i\lambda_{l_0+} = \lambda_{l_0+} \quad (62)$$

So the new transition from the base state to the new big peak state will be λ_{l_0+}.
To find the transition from the big peak state into the base state we use the relation from problem (2),

$$\lambda_{0peak} = \frac{\sum_{j=1}^{L}P_j\lambda_{l_i-}}{\sum_{j=1}^{L}P_j} = \lambda_{l_i-} \quad (63)$$

Using the relation between transition and probabilites in a two level system, we find that,

$$P_{l0} = \frac{l_{i-}}{l_{i-}+l_{0+}} = \frac{\frac{1}{e\mathcal{D}_0}}{\frac{1}{(1-e)\mathcal{D}_0}+\frac{1}{e\mathcal{D}_0}} = (1-e) \quad (64)$$

It is clear that the sum of probabilities of being in any peak state will be e due to condition of total probability equal to one. The particular probability in the original system for state i will be $\alpha_i e$ we can check this using the equilibrium equation for frequencies in a particular L_i,

$$\begin{aligned} P_{l_i}\lambda_{l_i} &= P_{l_0}\alpha_i\lambda_{l_0+} \\ P_{l_i} &= \frac{(1-e)\alpha_i e\mathcal{D}_0}{(1-e)\mathcal{D}_0} = \alpha_i e \end{aligned} \tag{65}$$

5. **Problem 5** :

Using binomial distribution we find that,

$$P_0 = 0.96^4 = 0.8493466 \tag{66}$$

$$P_1 = 4(0.96)^3(1-0.96) = 0.1415578 \tag{67}$$

$$P_2 = 6(0.96)^2(1-0.96)^2 = 0.0088474 \tag{68}$$

$$P_3 = 4(0.96)(1-0.96)^3 = 0.0002458 \tag{69}$$

$$P_4 = (1-0.96)^4 = 0.00000256 \tag{70}$$

The unserved energy $\mathcal{U}$ will be a function of a particular load level x in a period of 1 year,

$$\begin{aligned} \mathcal{U}(x) =& max\{0, x-200\}0.8493466 + max\{0, x-150\}0.1415578 + max\{0, x-100\}0.0088474 \\ &+ max\{0, x-50\}0.0002458 + max\{0, x\}0.00000256[MWyear] \end{aligned} \tag{71}$$