ECE 588 Problem Set 4 – Solutions

Problem 1

The equivalent load is defined by,

$$\underset{\sim}{L}'_k = \underset{\sim}{L}'_{k-1} - \underset{\sim}{A}_k \tag{1}$$

$$\underset{\sim}{L}'_0 = \underset{\sim}{L} \tag{2}$$

So to find the relation between $\mathcal{L}'$ and $\mathcal{L}$ will be necessary relate $\underset{\sim}{L}'_k$ with $\underset{\sim}{L}_k$. We start from definition (2) and adding $c_k - c_k$,

$$\begin{aligned}
\underset{\sim}{L}'_k &= \underset{\sim}{L}'_{k-1} - \underset{\sim}{A}_k + c_k - c_k && (3)\\
&= \underset{\sim}{L}'_{k-1} + \underset{\sim}{Z}_k - c_k && (4)\\
&= \underset{\sim}{L}'_{k-2} - \underset{\sim}{A}_{k-1} + c_{k-1} - c_{k-1} + \underset{\sim}{Z}_k - c_k && (5)\\
&= \underset{\sim}{L}'_{k-2} + \underset{\sim}{Z}_{k-1} + \underset{\sim}{Z}_k - c_{k-1} - c_k && (6)\\
&= \ldots + \ldots + \ldots + \ldots && (7)\\
&= \underset{\sim}{L}'_0 + \sum_{i=1}^{k} \underset{\sim}{Z}_i - \sum_{i=1}^{k} c_i && (8)
\end{aligned}$$

Using the fact that $\underset{\sim}{L}_k = \underset{\sim}{L}_{k-1} + \underset{\sim}{Z}_k$, the relation of $\underset{\sim}{L}'_k$ to $\underset{\sim}{L}_k$ is,

$$\underset{\sim}{L}'_k = \underset{\sim}{L}_k - C_k \tag{9}$$

So using the definition of $\mathcal{L}'$ we find,

$$\mathcal{L}'_k(x) = P(\underset{\sim}{L}'_k > x) = 1 - P(\underset{\sim}{L}'_k \leq x) = 1 - P(L_k \leq x + C_k) = P(L_k > x + C_k) = \mathcal{L}_k(x + C_k) \tag{10}$$

$$\boxed{\mathcal{L}'_k(x) = \mathcal{L}_k(x + C_k)} \tag{11}$$

Now we will derive convolution formula. The distribution of $\underset{\sim}{A}_k$ is,

$$f_{\underset{\sim}{A}_k}(x) = q_k \delta(x) + p_i \delta(x - c_k) \tag{12}$$

Then convolution formula,

$$\begin{aligned}
F_{\underset{\sim}{L}'_k}(x) &= \int_{-\infty}^{\infty} F_{\underset{\sim}{L}'_{k-1}}(x + y)\,[q_k \delta(y) + p_k \delta(y - c_k)]\,dy && (13)\\
&= q_k F_{\underset{\sim}{L}'_{k-1}}(x) + p_k F_{\underset{\sim}{L}'_{k-1}}(x + c_k) && (14)
\end{aligned}$$

Then,

$$\begin{aligned}
1 - F_{\underset{\sim}{L}'_k}(x) &= 1 - q_k F_{\underset{\sim}{L}'_{k-1}}(x) - p_k F_{\underset{\sim}{L}'_{k-1}}(x + c_k) && (15)\\
&= q_k + p_k - q_k F_{\underset{\sim}{L}'_{k-1}}(x) - p_k F_{\underset{\sim}{L}'_{k-1}}(x + c_k) && (16)\\
&= q_k \left(1 - F_{\underset{\sim}{L}'_{k-1}}(x)\right) + p_k \left(1 - F_{\underset{\sim}{L}'_{k-1}}(x + c_k)\right) && (17)
\end{aligned}$$

So, finally,

$$\boxed{\mathcal{L}_k'(x) = q_k \mathcal{L}_{k-1}'(x) + p_k \mathcal{L}_{k-1}'(x - c_k)} \tag{18}$$

Problem 2

Let's evaluate first $\mathcal{E}_k$. We know from lecture notes that in terms of $\mathcal{L}_k(x)$,

$$\mathcal{E}_k = p_k T \int_{C_{k-1}}^{C_k} \mathcal{L}_{k-1}(x) dx \tag{19}$$

From problem (1) we also know that,

$$\mathcal{L}_k'(x) = \mathcal{L}_k(x + C_k) \to \mathcal{L}_{k-1}(x) = \mathcal{L}_{k-1}'(x - C_{k-1}) \tag{20}$$

Put (20) into (19) we obtain,

$$\mathcal{E}_k = p_k T \int_{C_{k-1}}^{C_k} \mathcal{L}_{k-1}'(x - C_{k-1}) dx \tag{21}$$

We make a change of varibles $x - C_{k-1} = u \to dx = du$, and changing the limits of the integral, we obtain,

$$\boxed{\mathcal{E}_k = p_k T \int_0^{c_k} \mathcal{L}_{k-1}'(u) du} \tag{22}$$

In a similar way $\mathcal{U}_k$ in terms of $\mathcal{L}_k(x)$ is given by,

$$\mathcal{U}_k = T \int_{C_k}^{\infty} \mathcal{L}_k(x) dx \tag{23}$$

From (20) we have that $\mathcal{L}_k(x) = \mathcal{L}_k'(x - C_k)$ so in (23) we obtain,

$$\mathcal{U}_k = T \int_{C_k}^{\infty} \mathcal{L}_k'(x - C_k) dx \tag{24}$$

Performing a change of variables $x - C_k = u \to dx = du$, and changing the limits of the integral, we obtain,

$$\boxed{\mathcal{U}_k = T \int_0^{\infty} \mathcal{L}_k'(u) du} \tag{25}$$

Problem 3

We use the relation,

$$\mathcal{L}_{j-1}(x) = p_\alpha \mathcal{L}_I(x) + q_\alpha \mathcal{L}_I(x - c_i) \tag{26}$$

in the expresion that we have to check,

$$\mathcal{L}_I(x) = \frac{1}{p_\alpha} \left(\sum_{\nu=0}^{k-1} (-h)^\nu \mathcal{L}_{j-1}(x - \nu c_i) \right) + (-h)^k \tag{27}$$

where $(k-1)c_i < x \leq kc_i$.

Then in the RHS we get,

$$\frac{1}{p_\alpha}\left(\sum_{\nu=0}^{k-1}(-h)^\nu p_\alpha \mathcal{L}_I(x-\nu c_i) + (-h)^\nu q_\alpha(x-(\nu+1)c_i)\right) + (-h)^k \tag{28}$$

Put in a explicit way important terms $\nu = 0$ in the first part of summation and $\nu = k-1$ in the second part,

$$\mathcal{L}_I(x) + \sum_{\nu=1}^{k-1}(-h)^\nu \mathcal{L}_I(x-\nu c_i) + \sum_{\nu=0}^{k-2} h(-h)^\nu \mathcal{L}_I(x-(\nu+1)c_i) + \tag{29}$$

$$(h)(-h)^{k-1}\mathcal{L}_I(x-kc_i) + (-h)^k \tag{30}$$

Using the fact that $\mathcal{L}_I(x-kc_i) = 1$ in the set $(k-1)c_i < x \leq kc_i$, we can cancel the last two terms. We perform a change of variable in the second summation $\nu + 1 = \mu$ we get,

$$\mathcal{L}_I(x) + \sum_{\nu=1}^{k-1}(-h)^\nu \mathcal{L}_I(x-\nu c_i) - \sum_{\mu=1}^{k-1}(-h)^\mu \mathcal{L}_I(x-\mu c_i) = \boxed{\mathcal{L}_I(x)} \tag{31}$$

So we can see that the expresion is correct, because the summations cancels out (dummy indexes) so the RHS is $\mathcal{L}_I(x)$ like we expected.

Problem 4

From the relation $\underline{L}_{j-1} = \underline{I} + \underline{Z}_1$ the expression to show has to be (where we supposed that the first block of unit α is loaded in position $i-1$):

$$\mathcal{L}_{j-1}(x) = p_\alpha \mathcal{L}_{\underline{I}}(x) + q_\alpha \mathcal{L}_{\underline{I}}(x-c_1) \tag{32}$$

Let's work in the RHS using ,

$$\mathcal{L}_I(x) = \frac{1}{p_\alpha}\left(\sum_{\nu=0}^{k-1}(-h)^\nu \mathcal{L}_{j-1}(x-\nu c_i)\right) + (-h)^k \tag{33}$$

where $(k-1)c_i < x \leq kc_i$.

we obtain,

$$\sum_{\nu=0}^{k-1}\mathcal{L}_{j-1}(x-\nu c_i) + p_\alpha(-h)^k + \frac{q_\alpha}{p_\alpha}\sum_{\nu=0}^{k-1}(-h)^\nu \mathcal{L}_{j-1}(x-(\nu+1)c_i) + q_\alpha(-h)^k \tag{34}$$

Follow the same steps that problem (3), we expand explicitely some terms to see the cancelations,

$$\mathcal{L}_{j-1}(x) + \sum_{\nu=1}^{k-1}\mathcal{L}_{j-1}(x-\nu c_i) - \sum_{\nu=0}^{k-2}(-h)^{\nu+1}\mathcal{L}_{j-1}(x-(\nu+1)c_i) \tag{35}$$

$$-(-h)^k\mathcal{L}_{j-1}(x-kc_i) + (p_\alpha+q_\alpha)(-h)^k = \boxed{\mathcal{L}_{j-1}}$$

In this expresion we use $p_\alpha + q_\alpha = 1$, $\mathcal{L}_{j-1}(x-kc_i) = 1$ in this set and the same change of variables in summations used in problem (3), to obtain that the RHS is $\mathcal{L}_{j-1}$

Problem 5

From lecture notes the cost $\mathcal{C}_k$ as a function of the loaded capacity is,

$$\mathcal{C}_k(\mu) = p_k T \left(w_k(c_k^{min})\mathcal{L}_{k-1}[C_{k-1}] + \int_{C_{k-1}}^{C_{k-1}+\mu} w_k'[x - C_{k-1}]\mathcal{L}_{k-1} dx \right) \tag{36}$$

where $c_k \geq \mu \geq c_k^{min}$.

We also know that the expected energy in a range of energy $C_{k-1} < x < C_{k-1} + \mu$ is given by,

$$\mathcal{E}_k(\mu) = p_k T \int_{C_{k-1}}^{C_{k-1}+\mu} \mathcal{L}_{k-1}(x) dx \tag{37}$$

So if the marginal cost is constant and given by λ_β then $w_k'(x) = \lambda_\beta$ when $x > c_k^{min}$ and $w_k'(x) = 0$ when $0 < x \leq c_k^{min}$. Using this information about $w_k'(x)$ in (36) we can write the integral that $\int_{C_{k-1}}^{C_{k-1}+\mu} - \int_{C_{k-1}}^{C_{k-1}+c_k^{min}}$, so the cost is given by,

$$\mathcal{C}_k(\mu) = p_k T(w_k(c_k^{min})\mathcal{L}_{k-1}[C_{k-1}] + \lambda_\beta[\mathcal{E}_k(\mu) - \mathcal{E}_k(c_k^{min})] \tag{38}$$

Because in this case we have at least one block loaded, there is no constrain due to c_k^{min} because this was considered in the charge of the first block, with this in mind is clear that the extra cost of aditional blocks will be only the term $\lambda_\beta \mathcal{E}_k(\mu)$. In other words the integral

$$\int_{C_{k-1}}^{C_{k-1}+\mu} w_k'[x - C_{k-1}]\mathcal{L}_{k-1} dx \tag{39}$$

is split in two integrals, one for the first blocks and the additional,

$$\int_{C_{k-1}}^{C_{k-1}+\mu} w_k'[x-C_{k-1}]\mathcal{L}_{k-1}dx = \int_{C_{k-1}}^{C_{k-1}+\mu_{blocks1}} w_k'[x-C_{k-1}]\mathcal{L}_{k-1}dx + \int_{C_{k-1}+\mu_{blocks1}}^{C_{k-1}+\mu_{additional}} w_k'[x-C_{k-1}]\mathcal{L}_{k-1}dx \tag{40}$$

the first integral gives an expresion like (38) evaluate in $\mu = \mu_{blocks1}$. Because in *all* the interval $C_{k-1} + \mu_{blocks1} < x < C_{k-1} + \mu_{additional}$ the unitary cost is $w_k'(x) = \lambda_\beta$ we don't have the term c_k^{min} and the extra cost of this block will be $\lambda_\beta \mathcal{E}_k$.

Problem 6

The three state unit will have the distribution function,

$$f_{\underset{\sim}{A}_i}(x) = q_i\delta(x) + r_i\delta(x - d_i) + s_i\delta(x - c_i) \tag{41}$$

We use strategy similar to problem (1) of Problem Set 1, we can find the convolution formula in this case,

$$\underset{\sim}{M} = \sum_{j=1}^{i-1} \underset{\sim}{A}_j \tag{42}$$

and

$$\underset{\sim}{Y} = \underset{\sim}{A}_i \tag{43}$$

, then

$$P\{\sum_{l=1}^{i} \underset{\sim}{A}_l \leq x\} = F_{\underset{\sim}{M}+\underset{\sim}{Y}}(x) = \int_{-\infty}^{\infty} F_{\underset{\sim}{M}}(x - m) f_{\underset{\sim}{Y}}(m) dm \tag{44}$$

$$\boxed{P\{\sum_{l=1}^{i} \underset{\sim}{A}_l \leq x\} = q_i P\{\underset{\sim}{M} \leq x\} + r_i P\{\underset{\sim}{M} \leq x - d_i\} + s_i P\{\underset{\sim}{M} \leq x - c_i\}} \tag{45}$$

Now we can do a similar analysis to two level unit, in the context of production costing. Let's define the variable $\underset{\sim}{Z}_i = c_i - \underset{\sim}{A}_i$, this has the distribution given by,

$$f_{\underset{\sim}{Z}_i} = s_i \delta(x) + r_i \delta(x - (c_i - d_i)) + q_i \delta(x - c_i) \tag{46}$$

So to evaluate $\underset{\sim}{L}_i = \underset{\sim}{L}_{i-1} + \underset{\sim}{Z}_i$ we use convolution formula,

$$F_{\underset{\sim}{L}_i} = \int F_{\underset{\sim}{L}_{i-1}}(x - y) f_{\underset{\sim}{Z}_i}(y) dy \tag{47}$$

Using (46) in the last expresion we obtain,

$$1 - F_{\underset{\sim}{L}_i} = 1 - s_i F_{\underset{\sim}{L}_{i-1}}(x) - r_i F_{\underset{\sim}{L}_{i-1}}(x - (c_i - d_i)) - q_i F_{\underset{\sim}{L}_{i-1}}(x - c_i) \tag{48}$$

$$\mathcal{L}_i(x) = s_i \left[1 - F_{\underset{\sim}{L}_{i-1}}(x)\right] + q_i \left[1 - F_{\underset{\sim}{L}_{i-1}}(x - c_i)\right] + r_i \left[F_{\underset{\sim}{L}_{i-1}}(x - (c_i - d_i))\right] \tag{49}$$

So the expression in this case becomes,

$$\boxed{\mathcal{L}_i(x) = s_i \mathcal{L}_{i-1}(x) + q_i \mathcal{L}_{i-1}(x - c_i) + r_i \mathcal{L}_{i-1}(x - (c_i - d_i))} \tag{50}$$

For the energy, we take the relation given in lecture notes,

$$\mathcal{E}_i = \mathcal{U}_{i-1} - \mathcal{U}_i \tag{51}$$

$$= T \int_{C_{i-1}}^{\infty} \mathcal{L}_{i-1}(x) dx - T \int_{C_i}^{\infty} \mathcal{L}_i(x) dx \tag{52}$$

Using relation (50) into (52) we obtain,

$$\frac{\mathcal{E}_i}{T} = \int_{C_{i-1}}^{\infty} \mathcal{L}_{i-1}(x) dx - s_i \int_{C_i}^{\infty} \mathcal{L}_{i-1}(x) dx - r_i \int_{C_i}^{\infty} \mathcal{L}_{i-1}(x - (c_i - d_i)) dx - q_i \int_{C_i}^{\infty} \mathcal{L}_{i-1}(x - c_i) dx \tag{53}$$

$$= (s_i + r_i + q_i) \int_{C_{i-1}}^{\infty} \mathcal{L}_{i-1}(x) dx - s_i \int_{C_{i-1}+c_i}^{\infty} \mathcal{L}_{i-1}(x) dx - r_i \int_{C_{i-1}+d_i}^{\infty} \mathcal{L}_{i-1}(x) dx - q_i \int_{C_{i-1}}^{\infty} \mathcal{L}_{i-1}(x) dx \tag{54}$$

Where in the last line we made a change of variables in the integral. Simplifying a little bit we obtain,

$$\frac{\mathcal{E}_i}{T} = r_i \int_{C_{i-1}}^{C_{i-1}+d_i} \mathcal{L}_{i-1}(x) dx + s_i \int_{C_{i-1}}^{C_{i-1}+c_i} \mathcal{L}_{i-1}(x) dx \tag{55}$$

And split the last integral we obtain the expresion,

$$\boxed{\frac{\mathcal{E}_i}{T} = r_i \int_{C_{i-1}}^{C_{i-1}+d_i} \mathcal{L}_{i-1}(x) dx + s_i \int_{C_{i-1}}^{C_{i-1}+d_i} \mathcal{L}_{i-1}(x) dx + s_i \int_{C_{i-1}+d_i}^{C_i} \mathcal{L}_{i-1}(x) dx} \tag{56}$$