



Chapter 2: Mathematical Preliminaries

Mathematical Preliminaries for 2-D Image Reconstructions



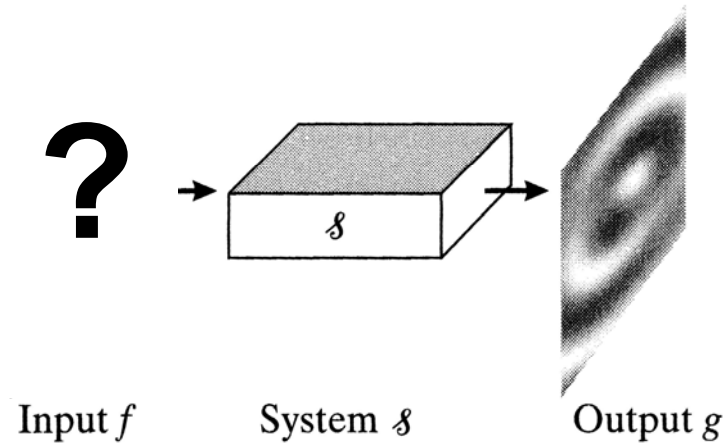
Contents

- Review of Fourier transform and filtering operations
- Projection data and imaging systems
- Radon transform
- Central slice theorem
- Sinogram
- Reconstruction with simple backprojection
- Filtered-backprojection
- Other analytical reconstruction methods.
- Matlab examples



The Inverse Problem

General Inverse Problem and Image Reconstruction



The inverse problem:

- Given a set of measured output signal
- Given the statistical description of the data
- Given a known system response function

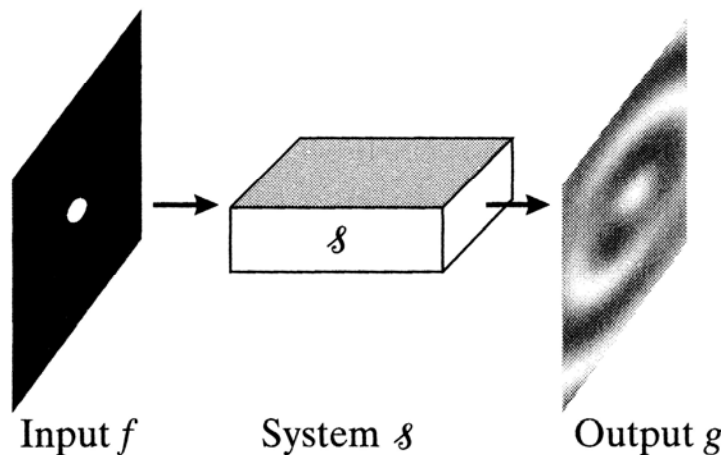
what should be the input signal that gave rise to the output data?

Convolution Theorem Revisited

The convolution theorem enables one to perform convolution operation as multiplication process in spatial frequency domain.

$$f(x, y) * g(x, y) = \mathcal{F}^{-1} \{ \mathcal{F}[f(x, y)] \cdot \mathcal{F}[g(x, y)] \}$$

By using the **Fast Fourier Transform** (FFT) algorithms, the convolution operation can be performed very efficiently !! This provide a practical way for modeling linear shift-invariant systems ...

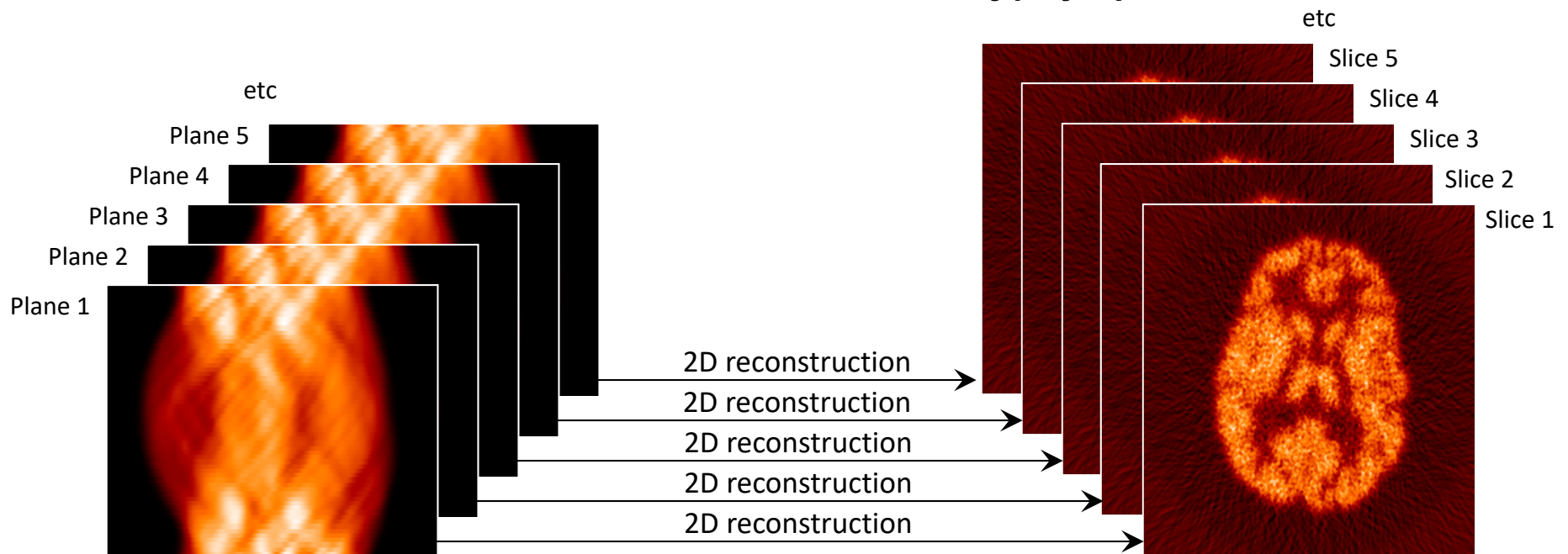


$$g(x, y) = f(x, y) * h(x, y)$$

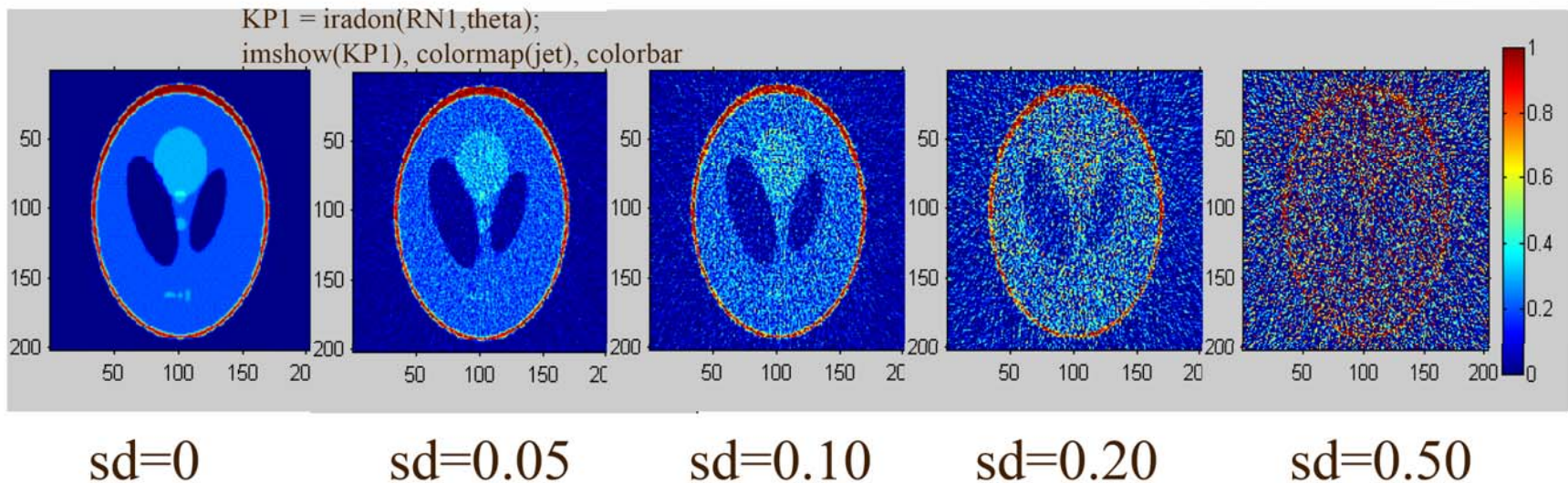
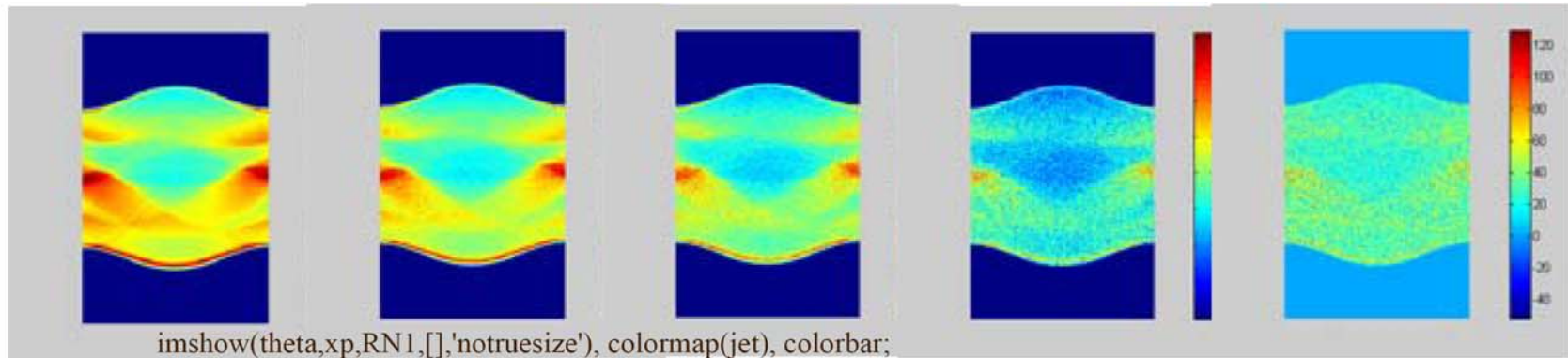
PET image reconstruction

- 2D Reconstruction

- Each parallel slice is reconstructed independently (a 2D sinogram originates a 2D slice)
- Slices are stacked to form a 3D volume $f(x,y,z)$



Matlab Examples



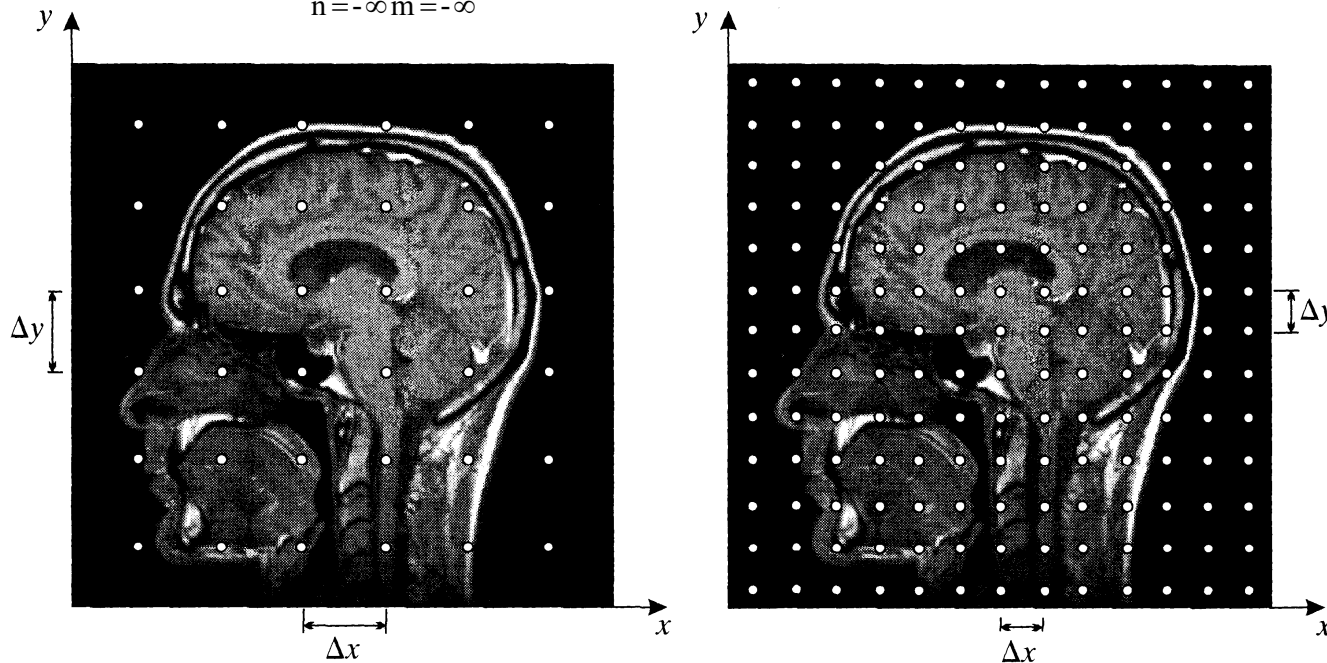


Sampling of a Signal

Review of Key Concepts

Two Dimensional Sampling

$$\begin{aligned}f_s(x, y) &= f(x, y) \cdot \delta_s(x, y, \Delta x, \Delta y) \\&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(x, y) \cdot \delta(x - n\Delta x, y - m\Delta y) \\&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(n\Delta x, m\Delta y) \cdot \delta(x - n\Delta x, y - m\Delta y)\end{aligned}$$



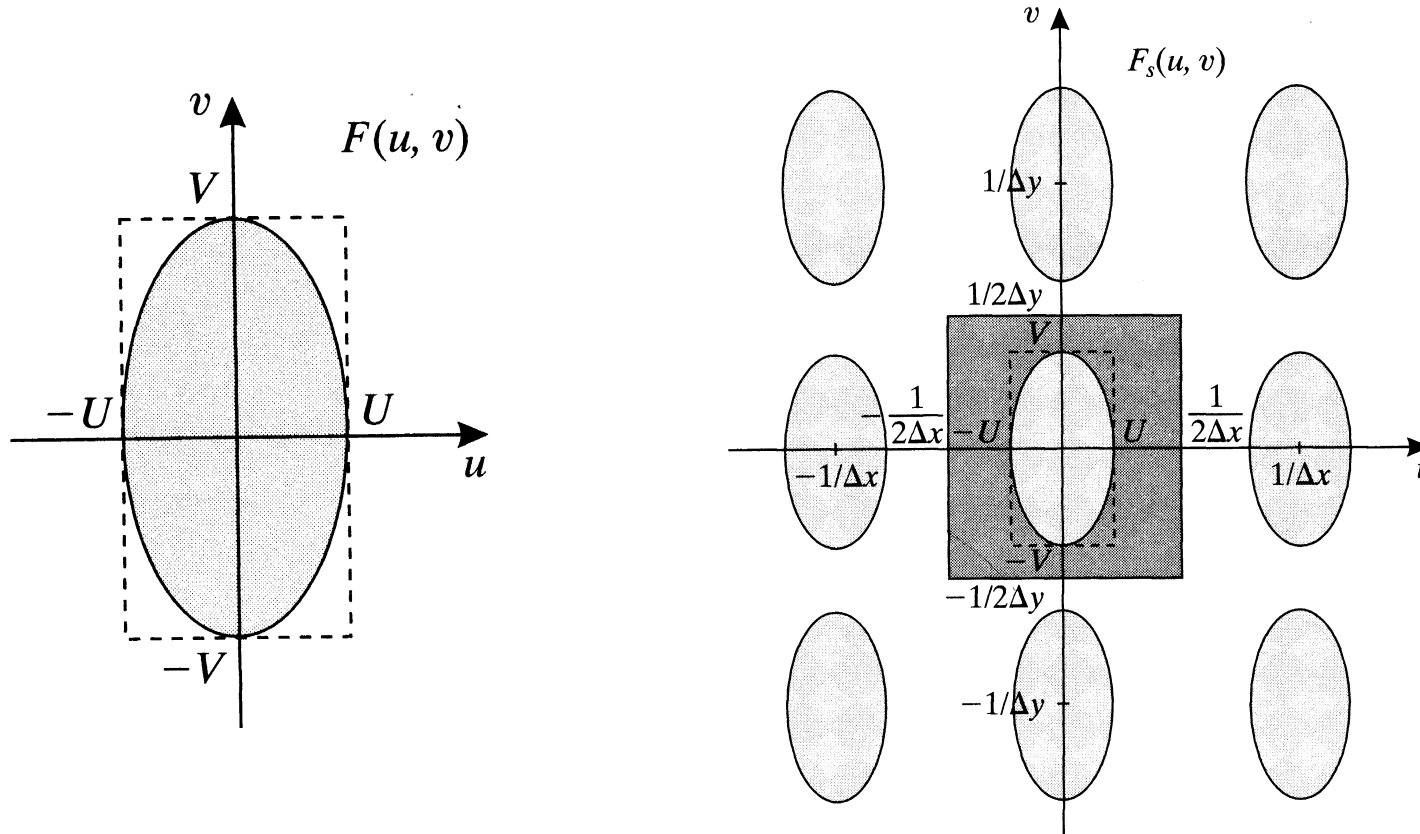
Fourier Transform of Sampled Image

$$\begin{aligned} F_{f_s}(u) &= \mathcal{F}[f_s(x, y)] \\ &= \mathcal{F}[\delta_s(x, y, \Delta x, \Delta y) \cdot f(x, y)] \\ &= \text{comb}(u \cdot \Delta x, v \cdot \Delta y) * \mathcal{F}[f(x, y)] \\ &= \sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} \delta \left[\Delta x \left(u - \frac{n}{\Delta x} \right), \Delta y \left(v - \frac{m}{\Delta y} \right) \right] * F(u, v) \\ &= \frac{1}{\Delta x \Delta y} \sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} \delta \left(u - \frac{n}{\Delta x}, v - \frac{m}{\Delta y} \right) * F(u, v) \\ &= \frac{1}{\Delta x \Delta y} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F \left(u - \frac{n}{\Delta x}, v - \frac{m}{\Delta y} \right) \end{aligned}$$

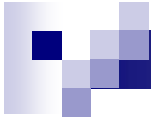
The result: Replicated $F(u, v)$, or “islands” every $1/\Delta x$ in u , and $1/\Delta y$ in v .

Review of Key Concepts (5)

Two Dimensional Sampling



An sufficiently small sampling interval is important ...



The Nyquist Theorem



Two Dimensional Sampling

Nyquist Theorem:

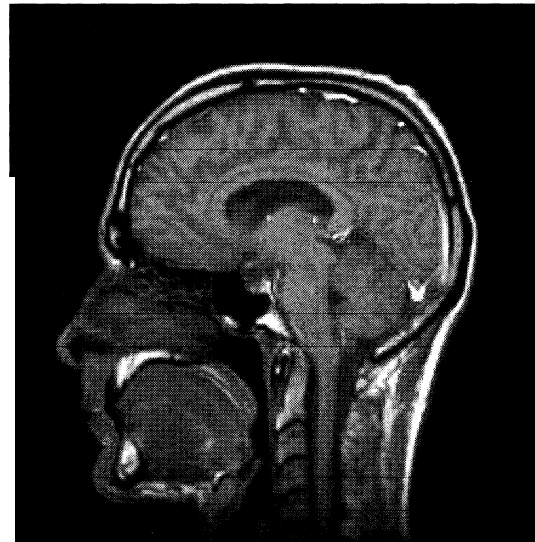
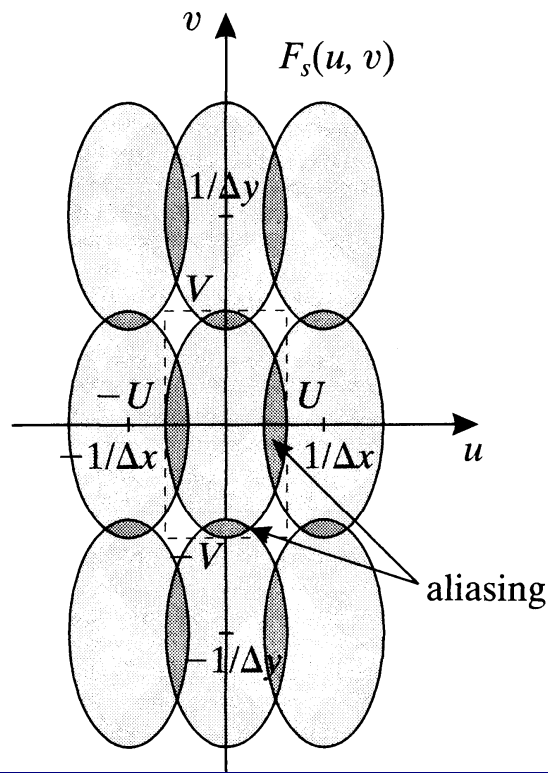
In order to restore the original function, the sampling rate must be greater than twice the highest frequency component of the function.

Nyquist Sampling Interval:

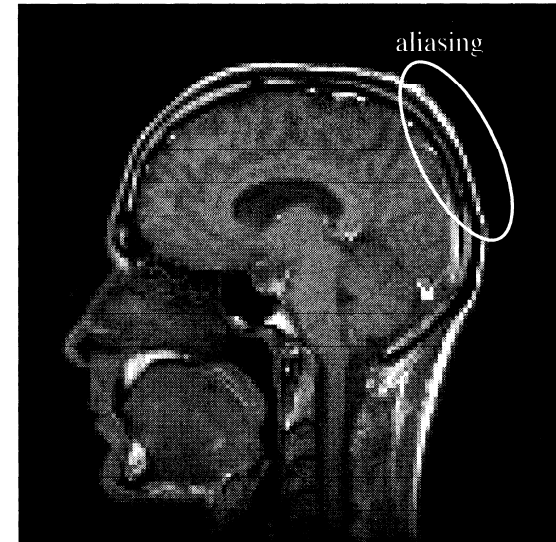
The maximum sampling interval allowed without introduce aliasing is

$$\Delta x \leq \frac{1}{2u_c}$$

Example – Fourier Transform of a Continuous Function



(a)



(b)

due to insufficient sampling



The Discrete Fourier Transform and the Equivalency of Spatial Domain and Frequency Domain Representations of a Signal



Review of Key Concepts

Discrete Fourier Transform

Nyquist sampling theorem indicates that all information contained in a continuous but band-width limited signal can be carried by just N samples.

Fourier transform is a no-loss transform.

So it make sense that we would only need a finite number (N) of Fourier coefficients to carry the same amount of information ...



Review of Key Concepts

Discrete Fourier Transform

The ***discrete Fourier transform*** (DFT) is defined as

$$F_n = \sum_{k=0}^{N-1} f_k e^{-\frac{j2\pi nk}{N}}, n = 0, 1, 2, \dots, N-1$$

$n = 0$ corresponding to the DC component (spatial frequency is zero)

$n = 1, \dots, N/2 - 1$ are corresponding to the positive frequencies $0 < u < u_c$

$n = N/2, \dots, N - 1$ are corresponding to the negative frequencies $-u_c < u < 0$

And the ***inverse DFT*** is defined as

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{\frac{j2\pi nk}{N}}, k = 0, 1, 2, \dots, N-1$$



Typical Filtering Operation in Fourier Domain

Review of Fourier Transform and Filtering

Spectral Filtering

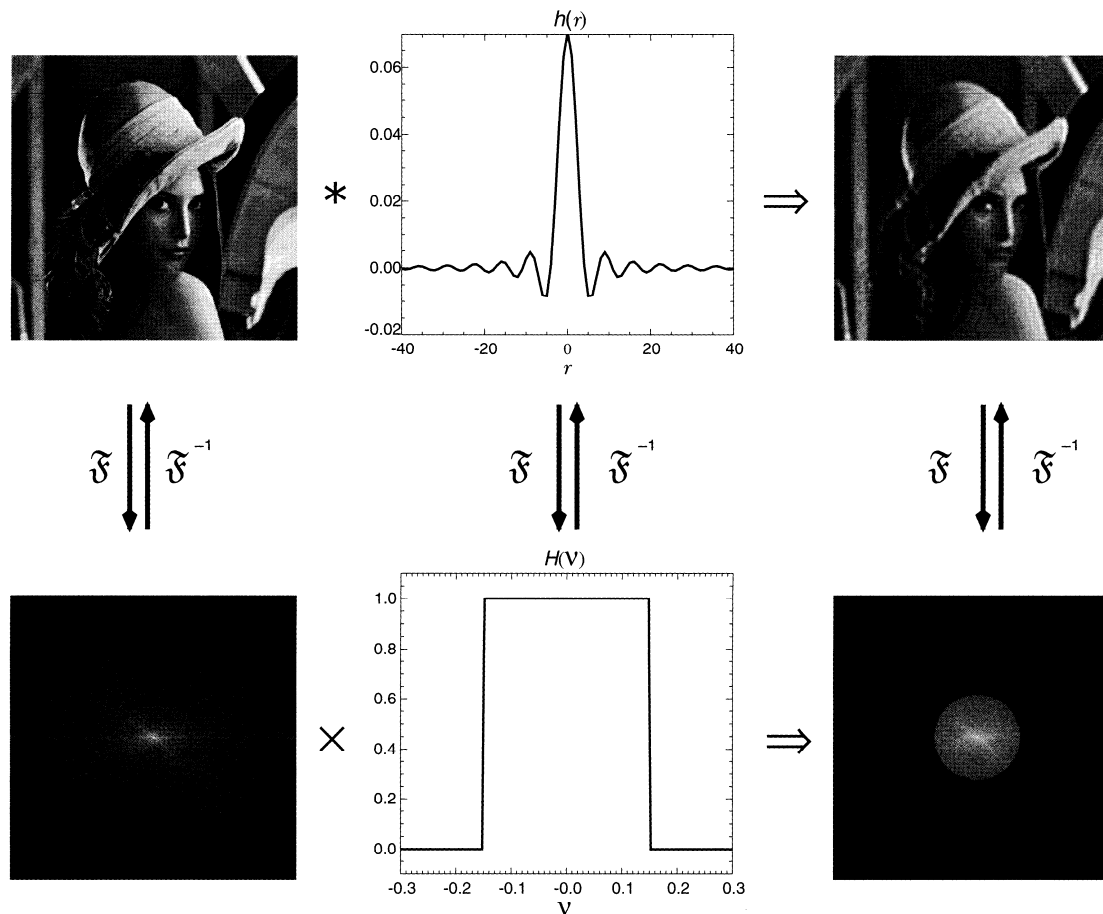


FIGURE 3 Ideal lowpass filtering of an image. Figures in the top and bottom rows demonstrate the filtering operation in the spatial and frequency domains, respectively. The left column shows the input image and the magnitude of its Fourier transform. The middle column are the circular symmetric PSF and its transfer function, where $v = \sqrt{v_x^2 + v_y^2}$ and $r = \sqrt{x^2 + y^2}$. Images in the right column are the output image and the magnitude of its Fourier transform.

Review of Fourier Transform and Filtering

Spectral Filtering

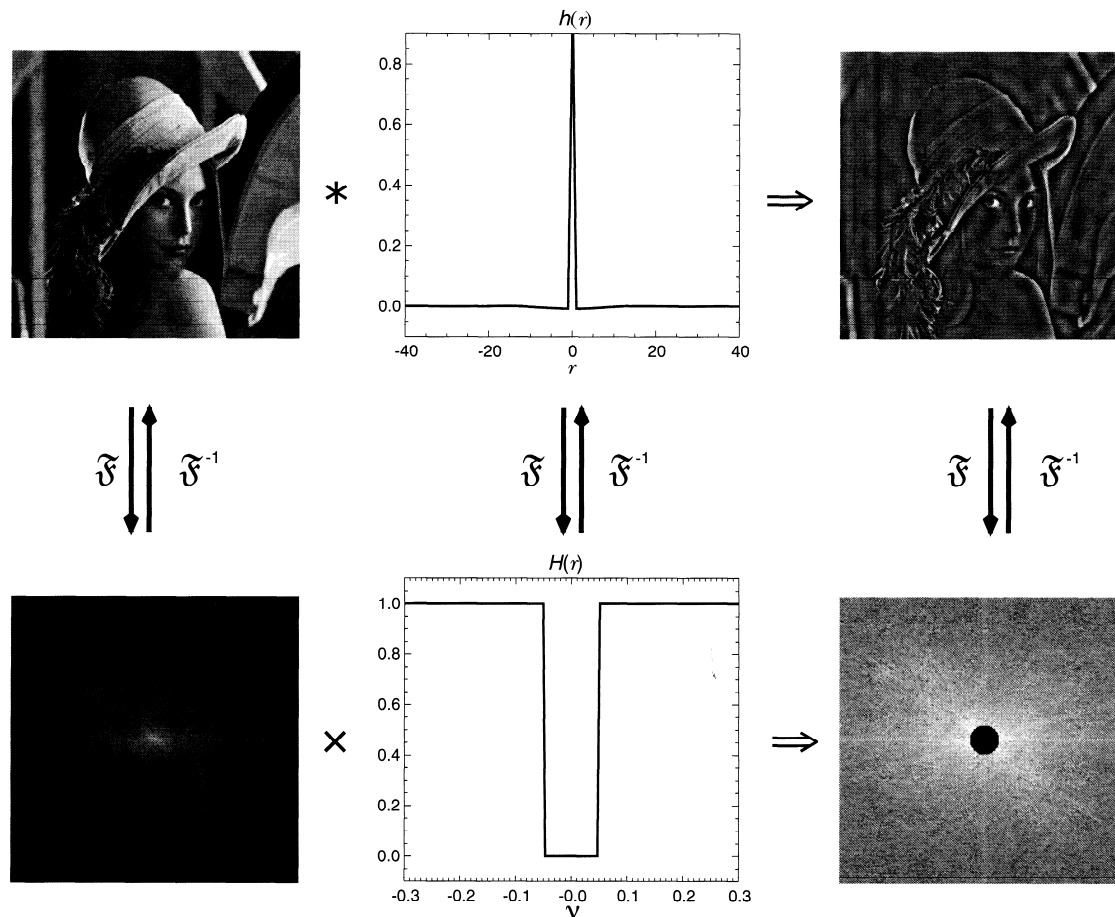


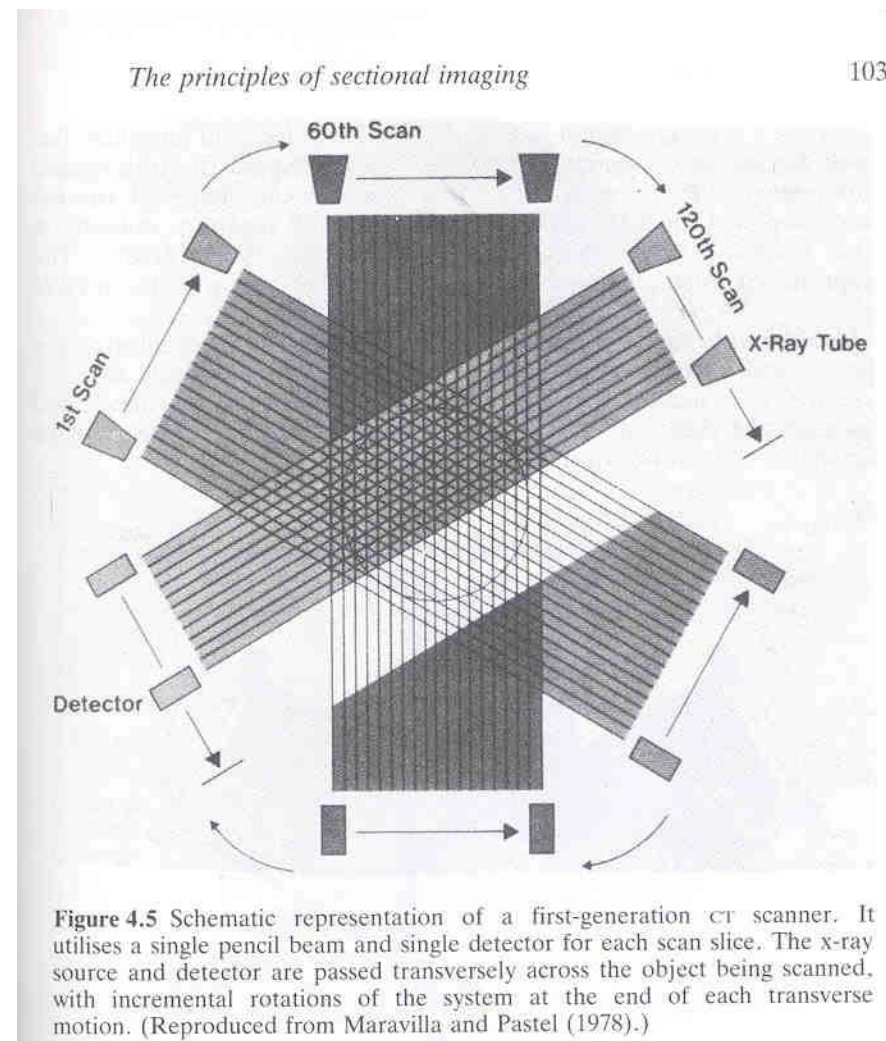
FIGURE 4 Ideal highpass filtering of an image. Figures in the top and bottom rows demonstrate the filtering operation in the spatial and frequency domains, respectively. The left column shows the input image and the magnitude of its Fourier transform. The middle column shows the circular symmetric PSF and its transfer function, where $v = \sqrt{v_x^2 + v_y^2}$ and $r = \sqrt{x^2 + y^2}$. Images in the right column are the output image and the magnitude of its Fourier transform.



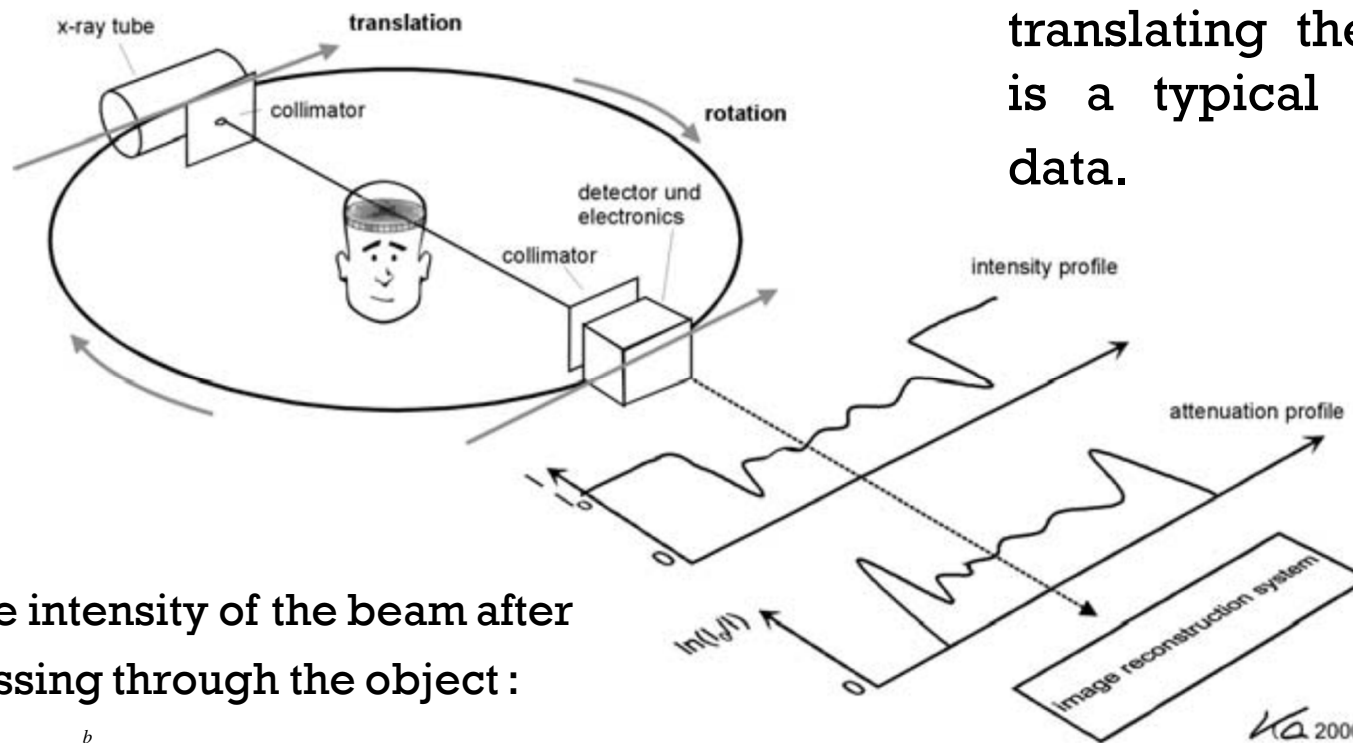
Projections

Projection Data from Early X-ray CT Systems

- Uses a collimator to keep exposure to a slice
- Builds image from multiple projections
- We will assume parallel rays for now.
 - Actually how first generation scanners worked.
 - Translate source and single detector across body for one angle
 - Then rotate source and detector to get next angle



Projection Data from Early X-ray CT Systems



Data measured by translating the detector is a typical projection data.

The intensity of the beam after passing through the object :

$$I = I_0 e^{-\int_a^b \mu(t) \cdot dt} \Leftrightarrow \ln(I_0 / I) = \int_a^b \mu(t) \cdot dt$$

From Computed Tomography, Kalender, 2000.

A Typical Gamma Camera

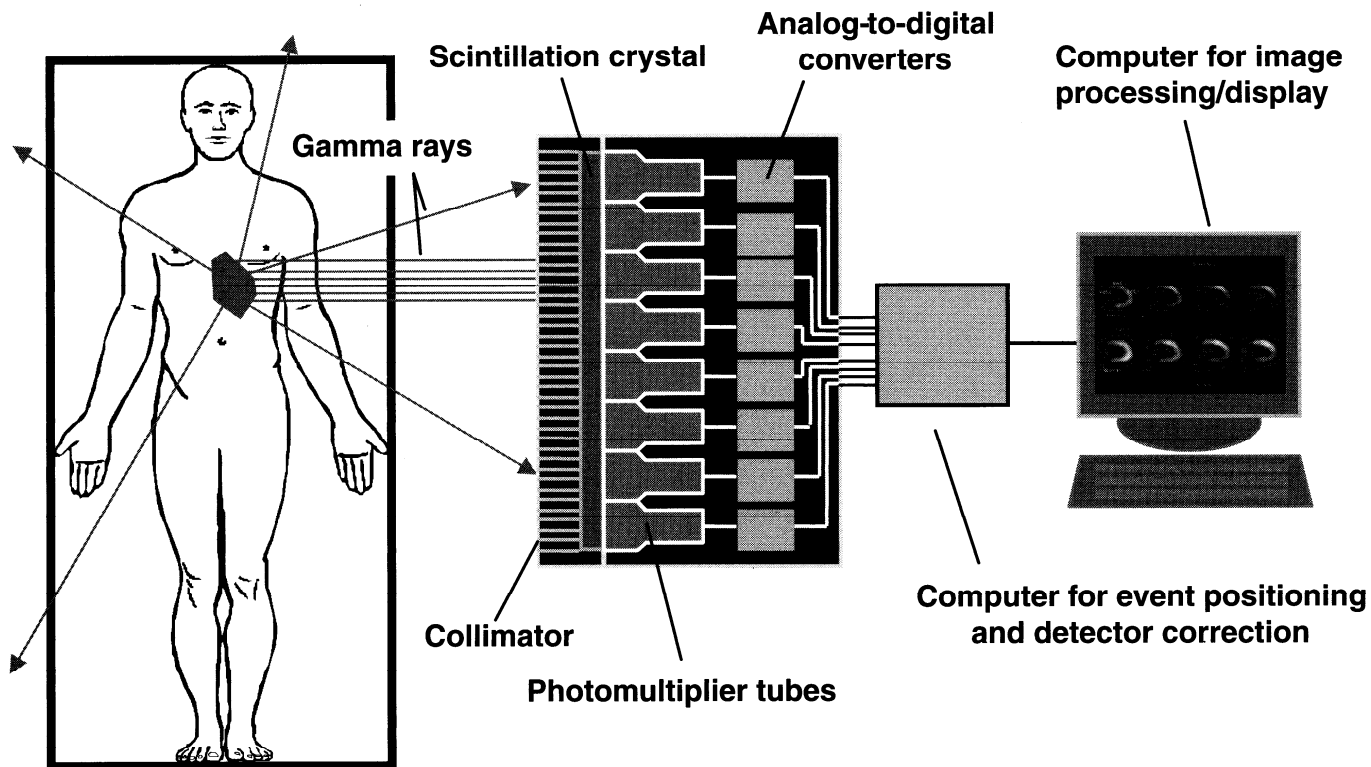
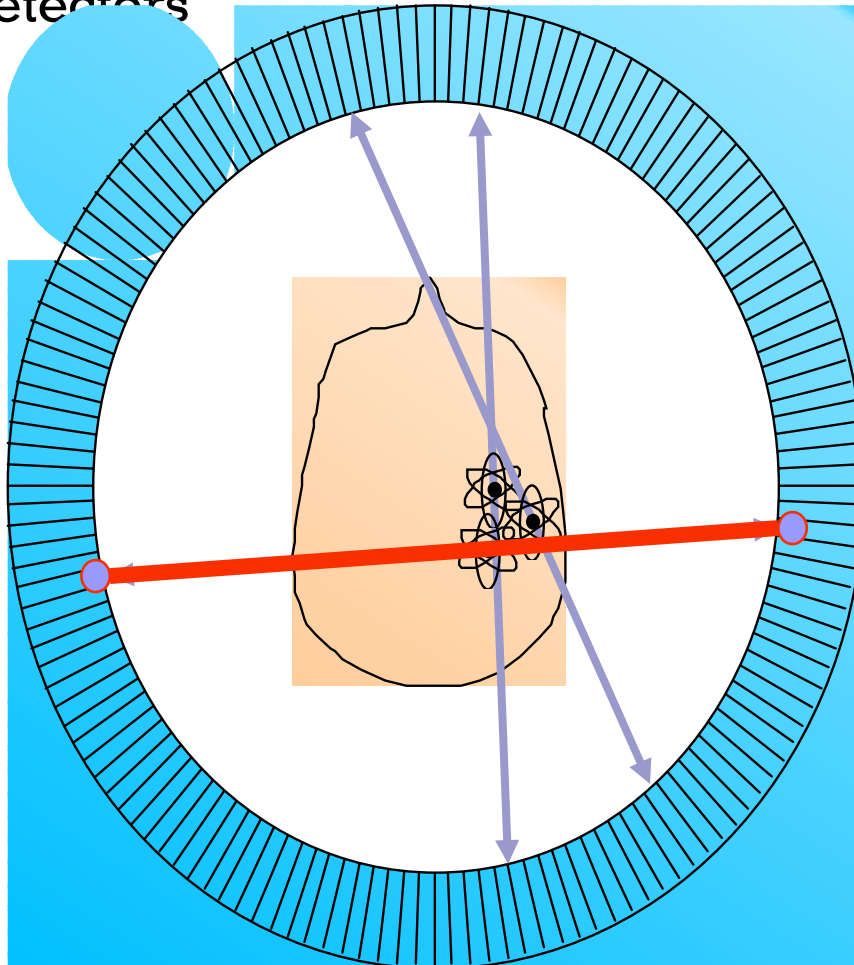


FIGURE 7 Schematic diagram of a conventional gamma camera used in SPECT. The collimator forms an image of the patient on the scintillation crystal, which converts gamma rays into light. The light is detected by photomultiplier tubes, the outputs of which are digitized and used to compute the spatial coordinates of each gamma-ray event (with respect to the camera face). A computer is used to process, store, and display the images.

Detect Radioactive Decays

Ring of Photon
Detectors



- **Radionuclide decays, emitting β^+ .**
- **β^+ annihilates with e^- from tissue, forming back-to-back 511 keV photon pair.**
- **511 keV photon pairs detected via time coincidence.**
- **Positron lies on line defined by detector pair (known as a *chord* or a *line of response* or a *LOR*).**

Detect Pair of Back-to Back 511 keV Photons

Example -- MRI of the Brain

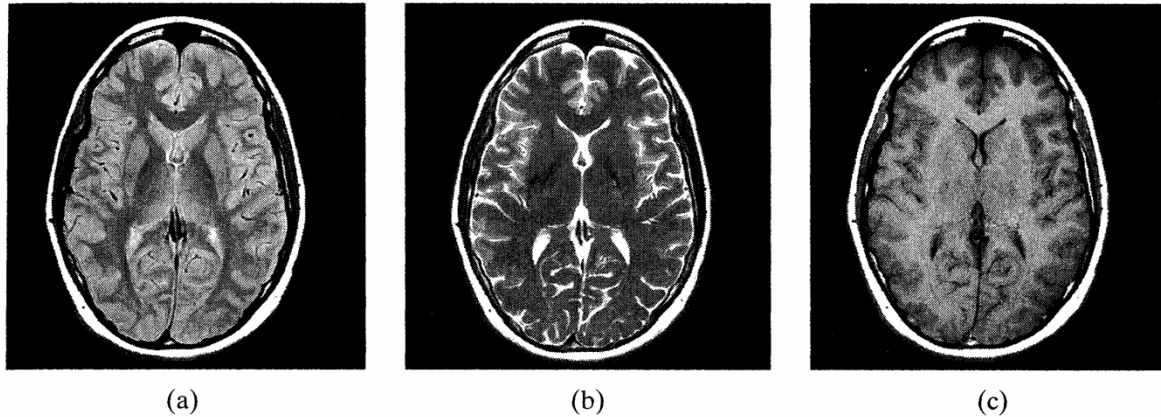


Figure 12.10 Three images of the same slice through the skull. Contrast between the tissue types are classified as (a) P_D -weighted, (b) T_2 -weighted, and (c) T_1 -weighted. (Used with permission of GE Healthcare.)

Proton Density
 $T_E = 17 \text{ ms}, T_R = 6000 \text{ ms}$

T_2 Contrast
 $T_E = 102 \text{ ms}, T_R = 6000 \text{ ms}$

T_1 Contrast
 $T_E = 17 \text{ ms}, T_R = 600 \text{ ms}$

- Long $T_R \rightarrow$ minimizing T_1 effect
- GM, WM decayed, CSF partially decayed \rightarrow large contrast between GM/WM and CSF
- Small contrast between GM and WM (WM is darker due to shorter T_2)

- Short $T_R \rightarrow$ building up T_1 effect
- T_R falls between T_1 's for GM, WM but much shorter than that for CSF \rightarrow GM/WM should build up more (brighter in the image) than CSF.

TABLE 12.2

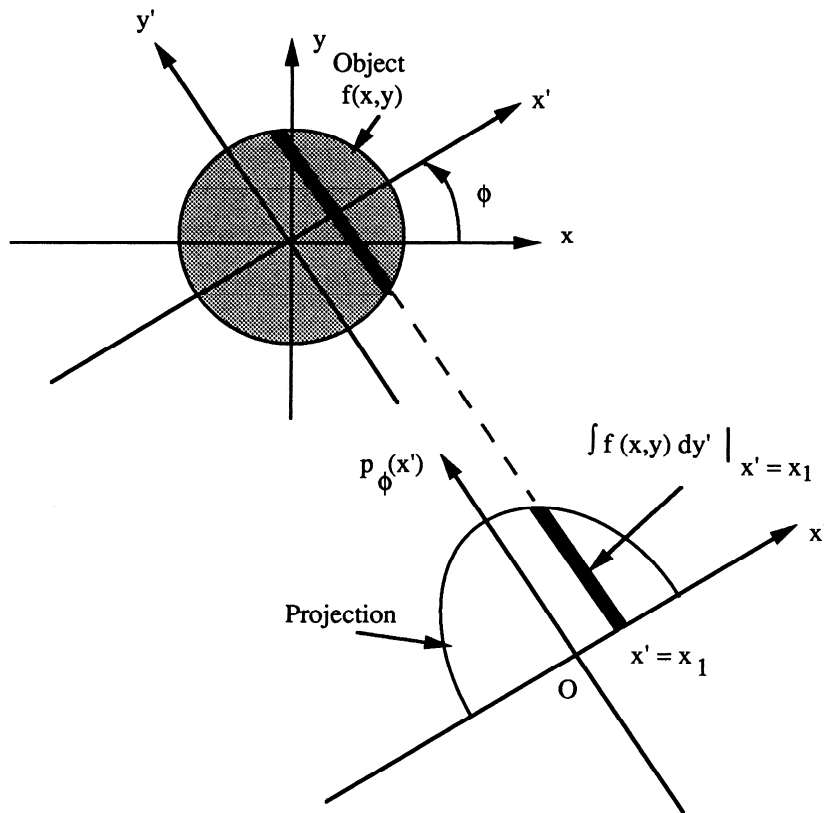
Typical Brain Tissue Parameters Measured at 1.5 T			
Tissue Type	Relative P_D	T_2 (ms)	T_1 (ms)
White matter	0.61	67	510
Gray matter	0.69	77	760
Cerebrospinal fluid	1.00	280	2650

Source: Adapted from Liang and Lauterbur, 2000.



How Do We Describe Projections in Math?

The Line Integral Projection

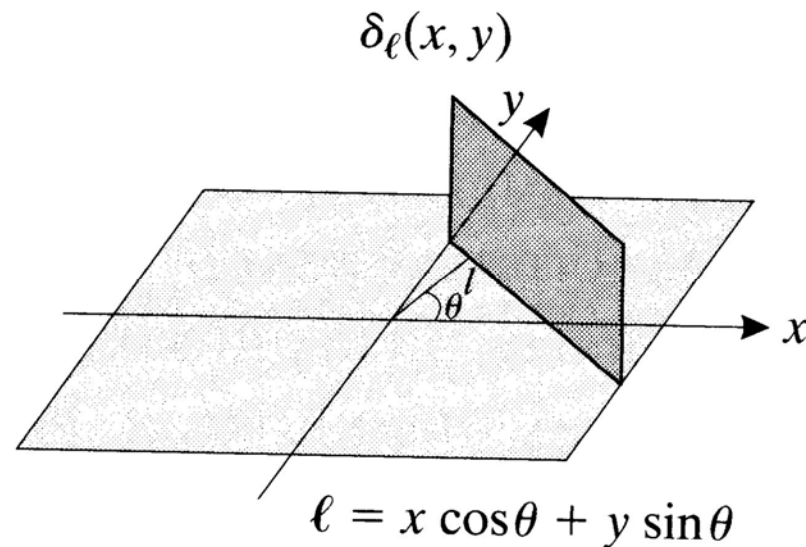


- Measures the integral property a physical object along a straight line.
- Projection data is naturally acquired by successive sampling the object at given angular intervals.

Line Impulse Signal (1)

$$\delta_L(x, y) = \delta(x \cos \theta + y \sin \theta - l)$$

$$\text{where } \delta(x) = \begin{cases} > 0, & x \cos \theta + y \sin \theta = l \\ 0, & \text{otherwise} \end{cases}$$



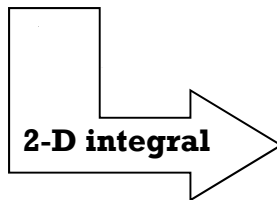
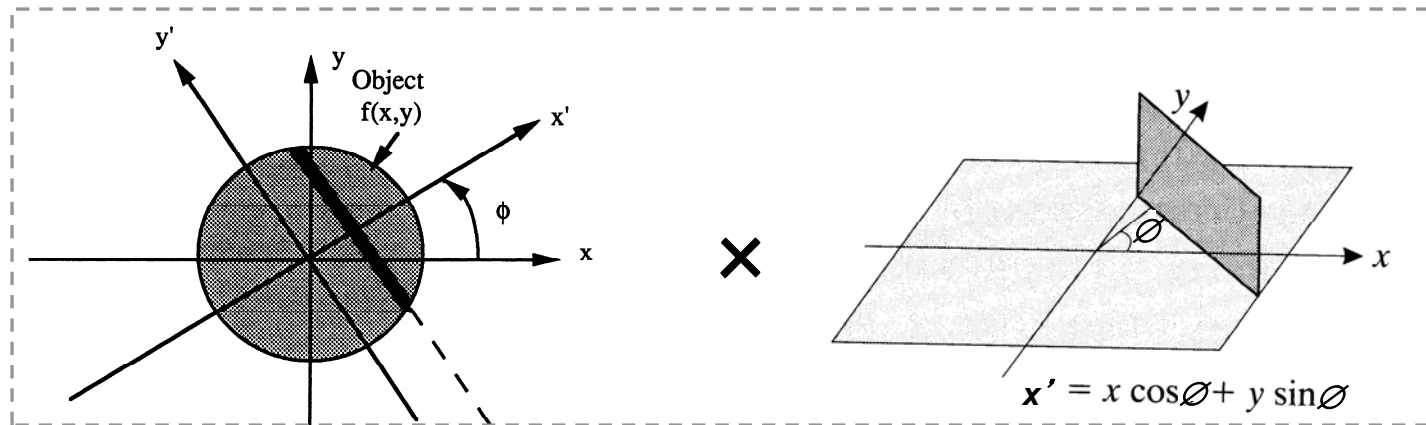


Line Impulse Signal (1)

$$\delta_L(x, y) = \delta(x \cos \theta + y \sin \theta - l)$$

- It can be used to measure the spatial resolution of a given imaging system.
- It is used to calculate the line-integral projection data for a given 2-D object.

Line Impulse Signal (2)



The value of the projection function $p_\phi(x')$ at this point is the integral of the function of $f(x,y)$ along the straight line: $x' = x \cos \phi + y \sin \phi$

The integral of a line impulse function and a given 2-D signal gives the **projection** data from a given view ...

$$p(\phi, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy$$

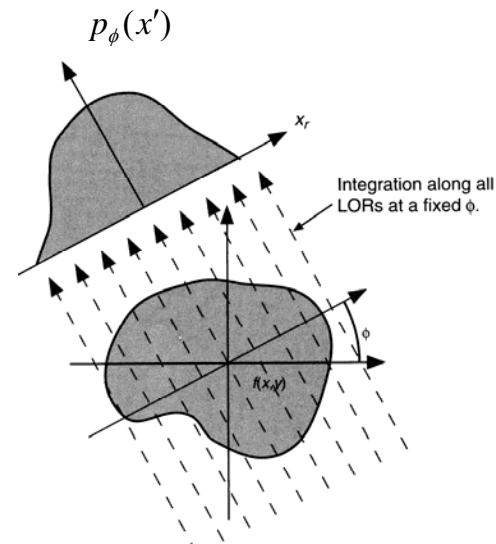
Radon Transform (1)

The Radon transform of a 2-D function is defined as

$$\begin{aligned} p(\phi, x') &\equiv \mathcal{R}[f(x, y)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy \\ &= \int_{-\infty}^{\infty} f(x' \cos \phi - y' \sin \phi, x' \sin \phi + y' \cos \phi) dy' \end{aligned}$$

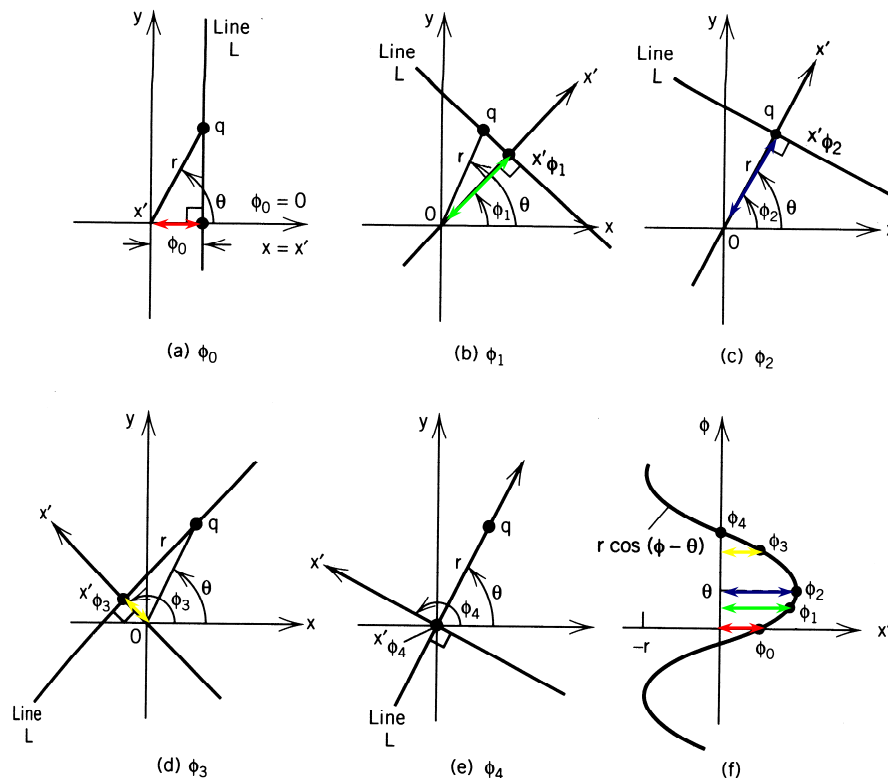
where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Radon Transform and Sinogram

- Radon transform maps a 2-D function $f(x,y)$ into a sinogram.

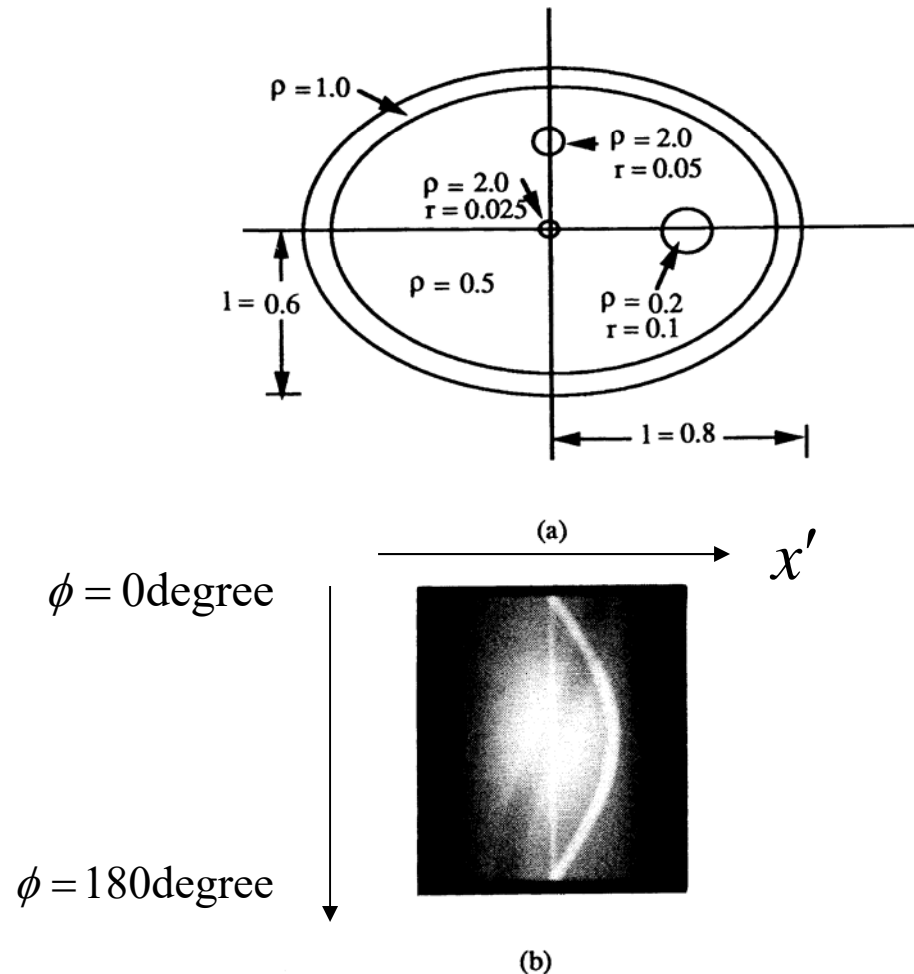


$$\mathcal{P}(\phi, x') = \mathcal{R}[f(x, y)]$$

Any point represented by polar coordinates (r, θ) will simply follow the equation $x' = r \cos(\phi - \theta)$ in the sinogram space.

Figure 3-2 Sinogram of a point q at a various angles ϕ_i for a given θ : (a) $\phi_0 = 0$, (b) $\phi_1 = 45^\circ$, (c) $\phi_2 = 60^\circ$, (d) $\phi_3 = 135^\circ$, and (e) $\phi_4 = 150^\circ$. At $i=0$ or $\phi_i=0$, the rotated coordinates x' coincide with the reference x -axis and give $x' = r \cos(\phi - \theta) = r \cos \theta$.

Radon Transform and Sinogram

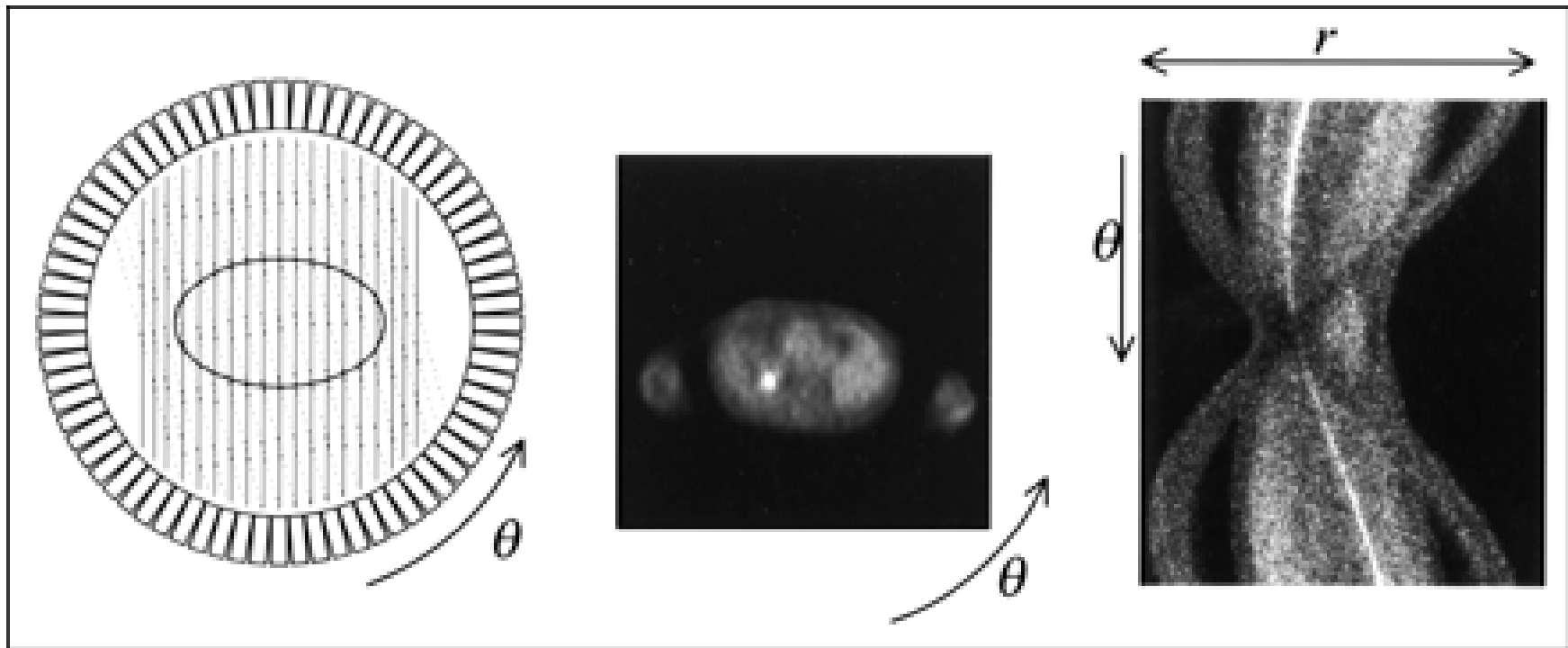


Sinogram is a 2-D function representing the original function $f(x,y)$ into the projection data space.

Sinogram is the basic data format for reconstruction

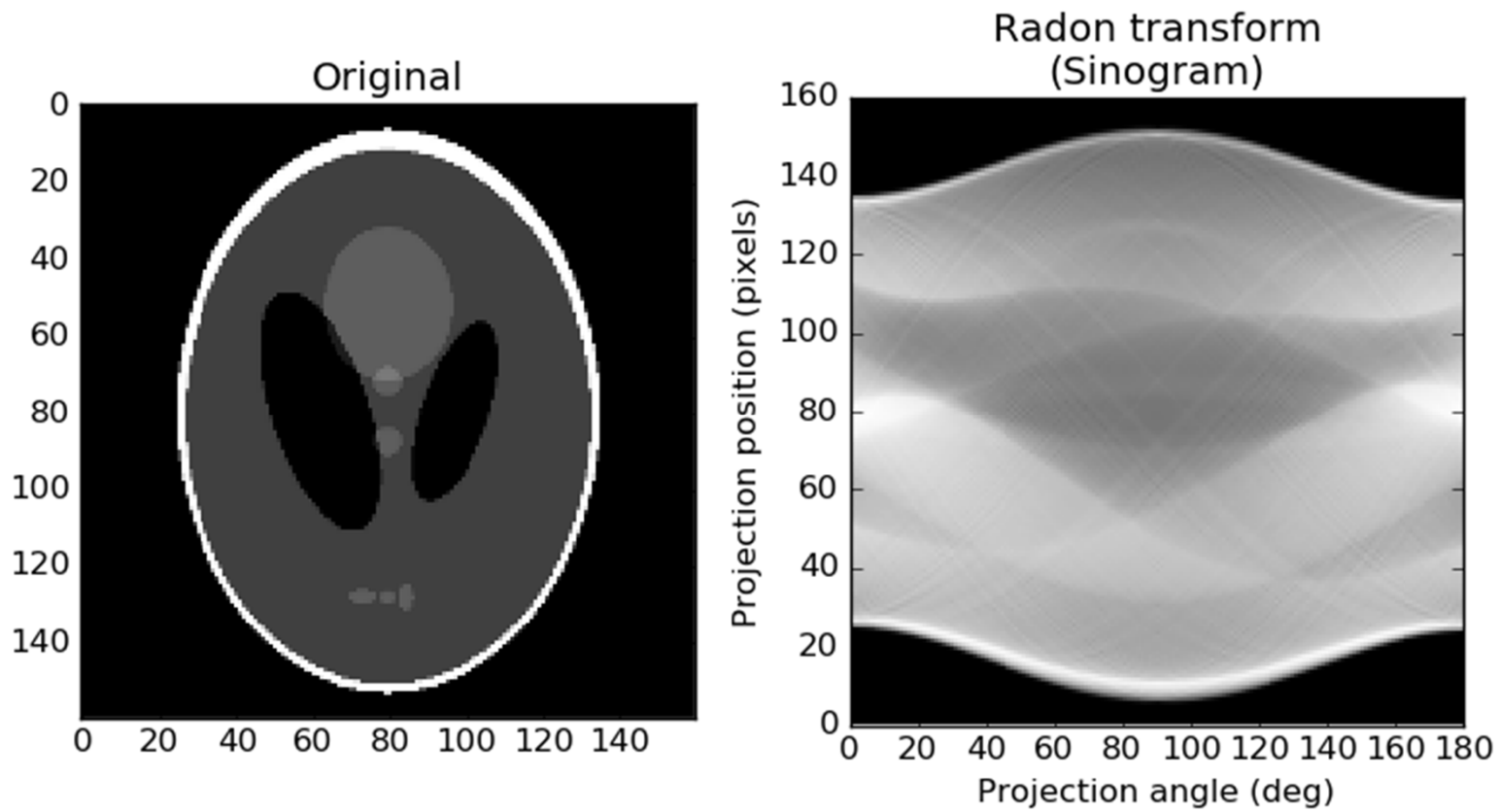
Figure 3-3 (a) Phantom with several distinct objects at various positions; (b) corresponding sinogram.

Radon Transform and Sinogram



<http://tech.snmjournals.org/cgi/content-nw/full/29/1/4/F3>

Radon Transform and Sinogram



<http://tech.snmjournals.org/cgi/content-nw/full/29/1/4/F3>



Radon Transform and Sinogram

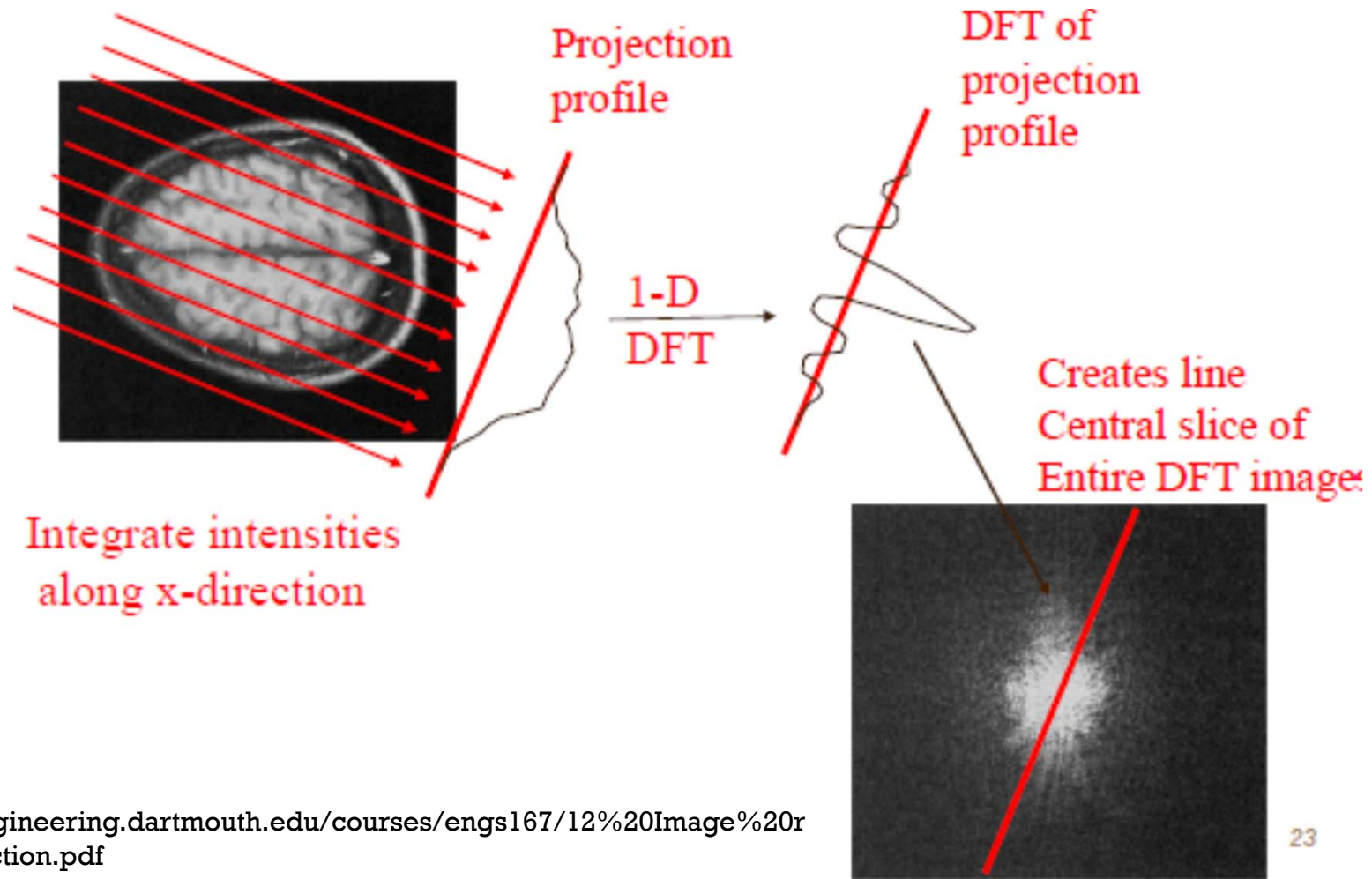
- Is Radon transform a no-loss transform?
- Can we restore the original function $f(x,y)$ from the sinogram?
- Under what condition that the $f(x,y)$ can be reconstructed exactly?
- How to perform the reconstruction?



What Exactly Does Projection Data Tell Us?

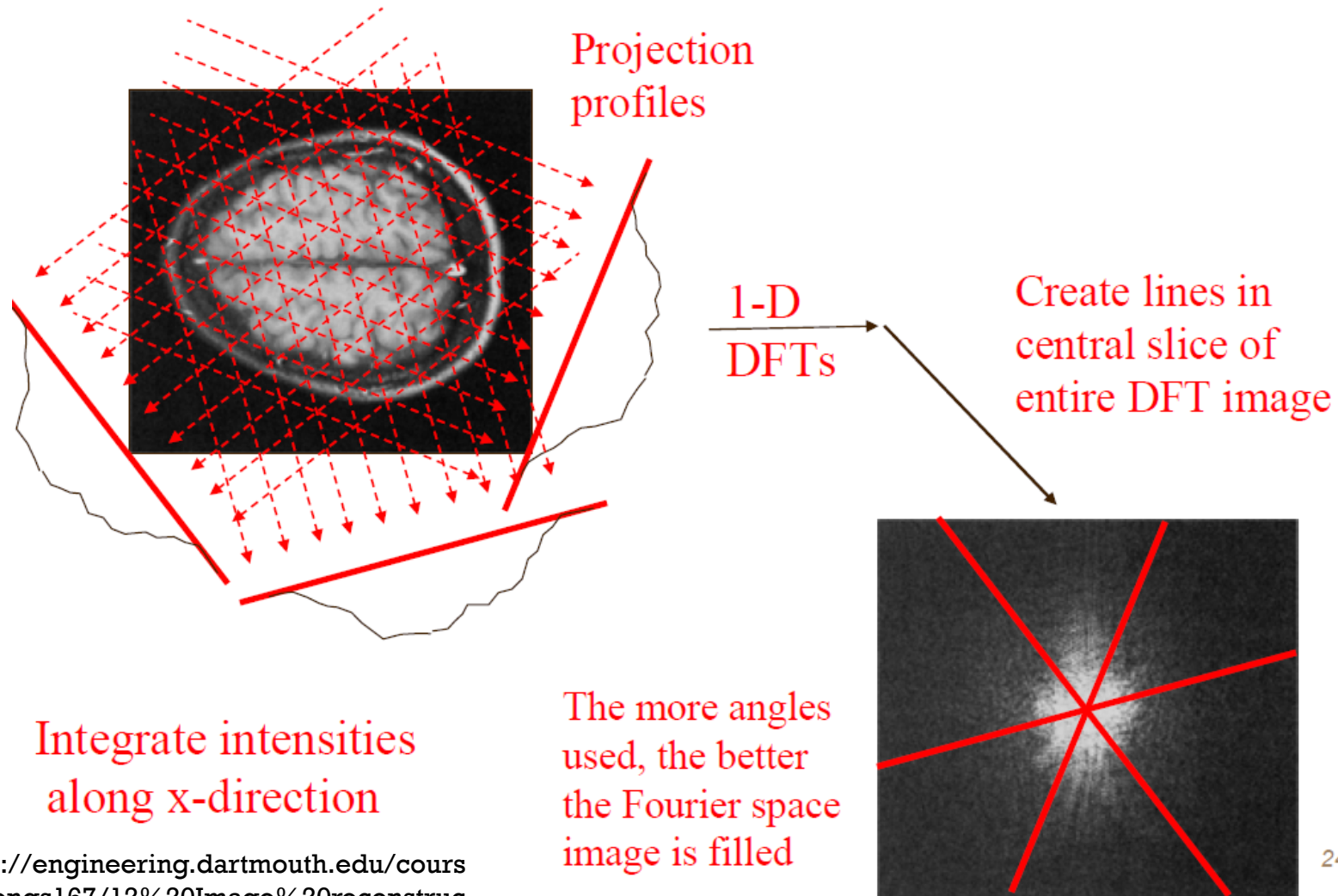
Central Slice Theorem

$$F\{p(\phi, \mathbf{x}')\} = F(r, \phi)$$



<http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf>

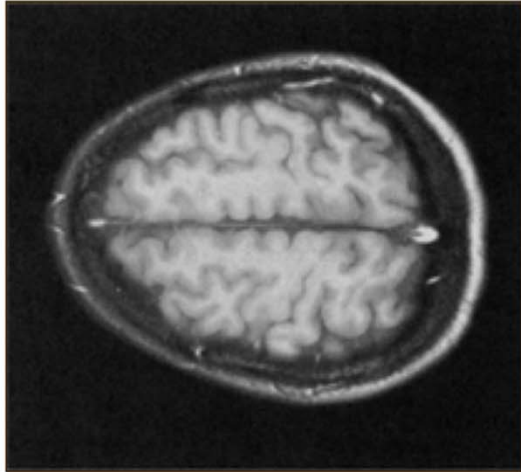
Central Slice Theorem



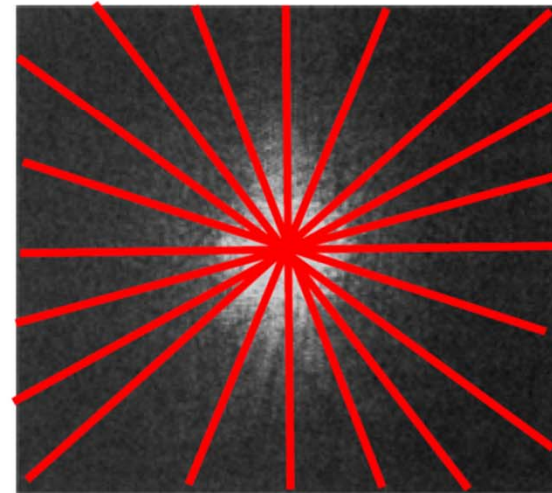
<http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf>

Central Slice Theorem

DFT image represents integration of original projections DFT transformed and summed together.



1-D
DFTs
at each
projection



This is the fast way to create the DFT image from projection data. The more projections taken, the more complete the sampling.

<http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf>

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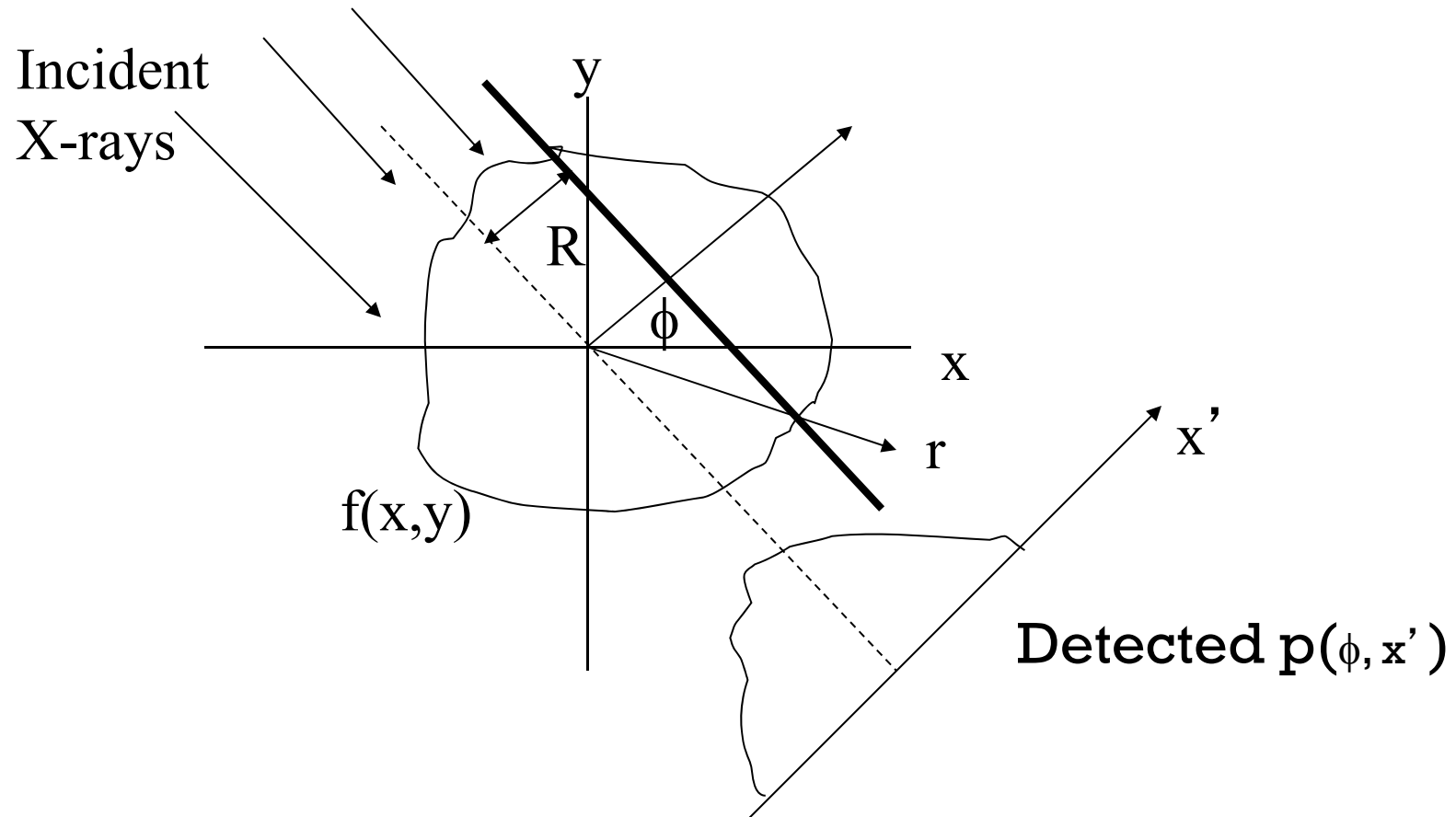
Central Section Or Projection Slice Theorem

$$F\{p(\phi, \mathbf{x}')\} = F(r, \phi)$$

So in words, the Fourier transform of a projection at angle ϕ gives us a line in the polar Fourier space at the same angle ϕ .

Central slice theorem is the key to understand reconstructions from projection data

Central Slice Theorem



The thick line is described by

$$x \cos \phi + y \sin \phi = R$$



Central Slice Theorem

The projection $g_{\phi}(x')$ can thus be calculated as a set of line integrals, each at a unique x' .

$$p(\phi, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy$$

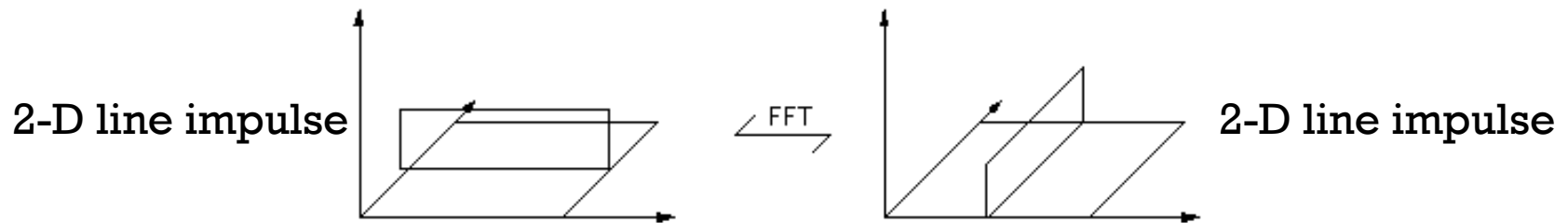
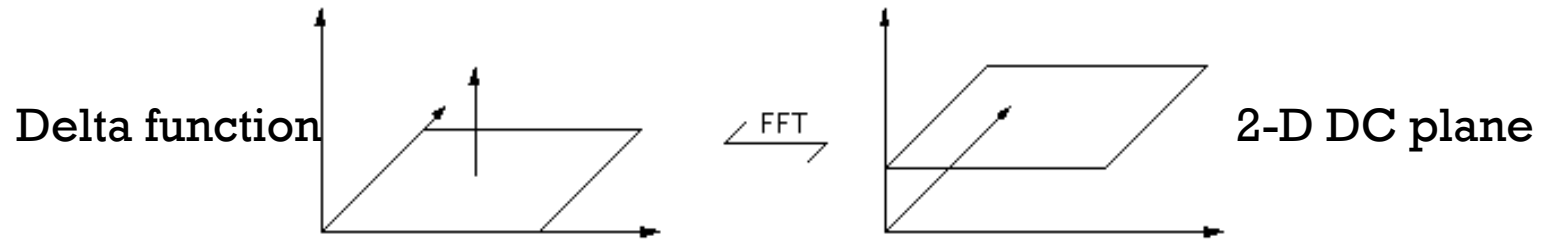
Or alternatively,

$$p(\phi, x') = \int_0^{\infty} \int_0^{2\pi} f(r, \theta) \delta(r \cos(\phi - \theta) - x') r dr d\theta$$

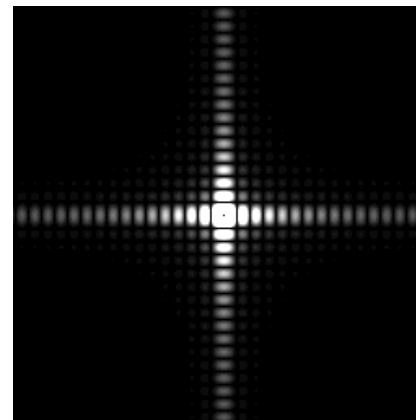
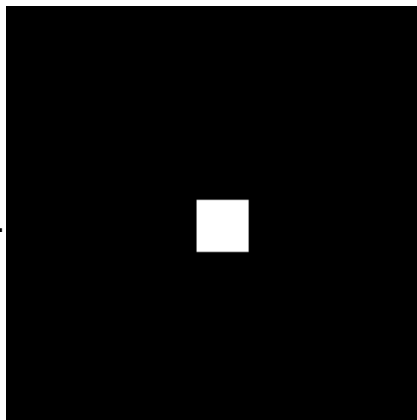
in polar coordinates.

Again $p(\phi, x')$ can be treated as a 1D function of x' at a given angle ϕ .

Examples



Square signal

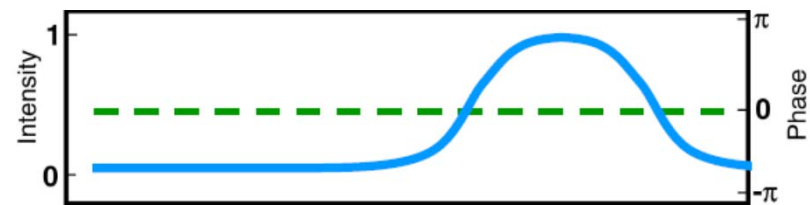


2-D sinc
function

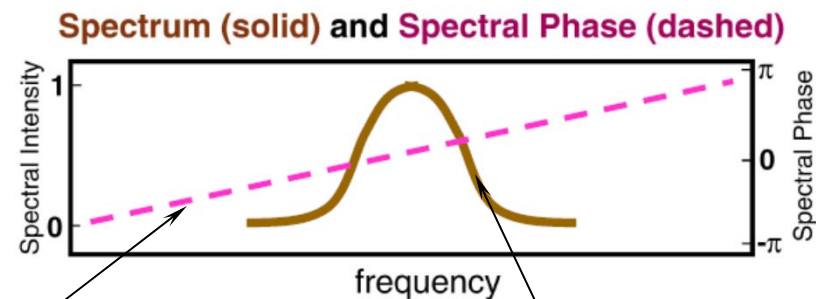
Properties of Fourier Transform

$$\mathcal{F}[f(x - a)] = \mathcal{F}[f(x)]e^{-j2\pi \cdot \omega \cdot a}$$

Spatial Domain



Spatial Frequency Domain



Linear shifting in spatial domain simply adds some linear phase to the pulse

Magnitude is unchanged

Central Slice Theorem

Let's consider the 2D FFT of an arbitrarily given function

$$F(u,v) = \iint f(x,y) e^{-i 2\pi (ux + vy)} dx dy$$

$$\begin{aligned} \mathcal{F}[f(x-t)] &= \mathcal{F}[f(x)] \cdot e^{-2\pi j \cdot \omega \cdot t} \\ \mathcal{F}[\delta(x)] &= 1 \\ \mathcal{F}[\delta(x-t)] &= e^{-2\pi j \cdot \omega \cdot t} \end{aligned}$$

In polar coordinates within the spatial-frequency domain,

$$u = \rho \cos \beta \qquad v = \rho \sin \beta$$

$$F(\rho, \beta) = \iint f(x,y) \exp[-i 2\pi \rho (x \cos \beta + y \sin \beta)] dx dy$$

If $x \cos \beta + y \sin \beta = \text{constant}$, $\exp[-i 2\pi \rho (x \cos \beta + y \sin \beta)]$ is a linear phase shift. This is the Fourier transform of a shifted delta function. Let's write the complex exponential as the FT of a δ function.

$$F(\rho, \beta) = \int_y \int_x f(x,y) F[\delta(x \cos \beta + y \sin \beta - x')] dx dy$$

$$F(\rho, \beta) = \int_y \int_x f(x,y) \int [\delta(x \cos \beta + y \sin \beta - x')] e^{-i 2\pi \rho x'} dx' dx dy$$

Central Slice Theorem

$$F(\rho, \phi) = \iint f(x, y) \int [\delta(x \cos \phi + y \sin \phi - x')] e^{-i 2\pi \rho x'} dx' dx dy$$

Recall how we wrote the projection as a double integral of $f(x, y)$ where a delta function performs the line integral,

$$p(\phi, x') = \iint f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy$$

We take the Fourier Transform of $p(\phi, x')$:

$$F\{p(\phi, x')\} = \int_{x'} \left[\int_y \int_x f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy \right] e^{-i 2\pi \rho x'} dx'$$

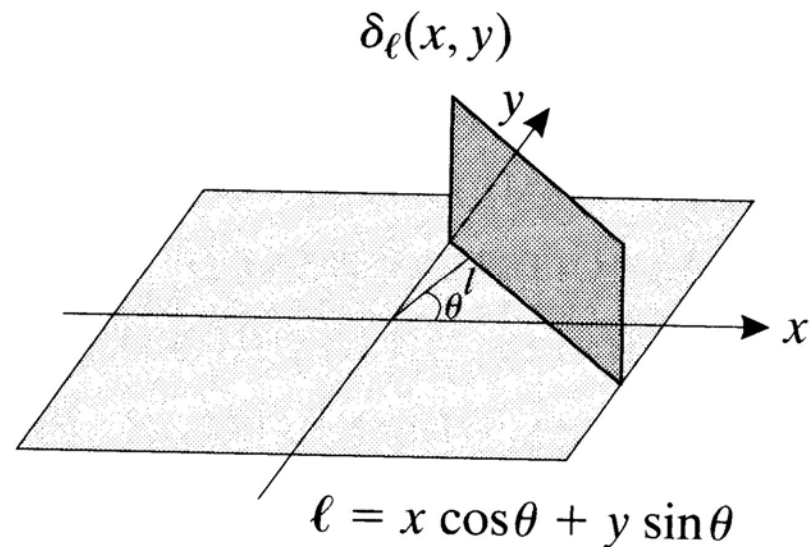
which is exactly what we wrote for $F(\rho, \beta)$ above!

$$\text{Thus, } F\{p_\phi(x')\} = F(\rho, \phi)$$

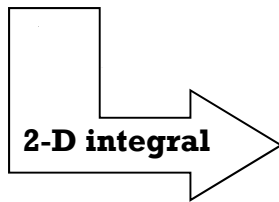
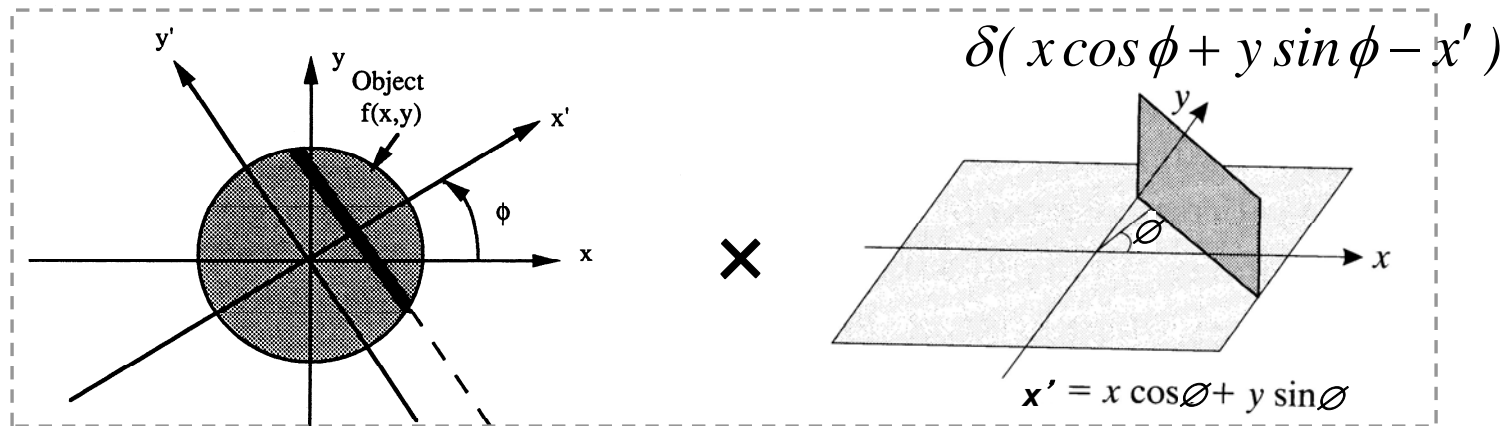
Line Impulse Signal (1)

$$\delta_L(x, y) = \delta(x \cos \theta + y \sin \theta - l)$$

$$\text{where } \delta(x) = \begin{cases} > 0, & x \cos \theta + y \sin \theta = l \\ 0, & \text{otherwise} \end{cases}$$



Line Impulse Signal (2)



The value of the projection function $p_\phi(x')$ at this point is the integral of the function of $f(x,y)$ along the straight line: $x' = x \cos \phi + y \sin \phi$

The integral of a line impulse function and a given 2-D signal gives the **projection** data from a given view ...

$$p(\phi, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy$$



Projection Theorem or Central Slice Theorem

An alternative proof:

See Page 77, Foundations of Medical Imaging, Z. H. Cho.



Central Section Or Projection Slice Theorem

$$F\{p(\phi, \mathbf{x}')\} = F(r, \phi)$$

So in words, the Fourier transform of a projection at angle ϕ gives us a line in the polar Fourier space at the same angle ϕ .

Central slice theorem is the key to understand reconstructions from projection data