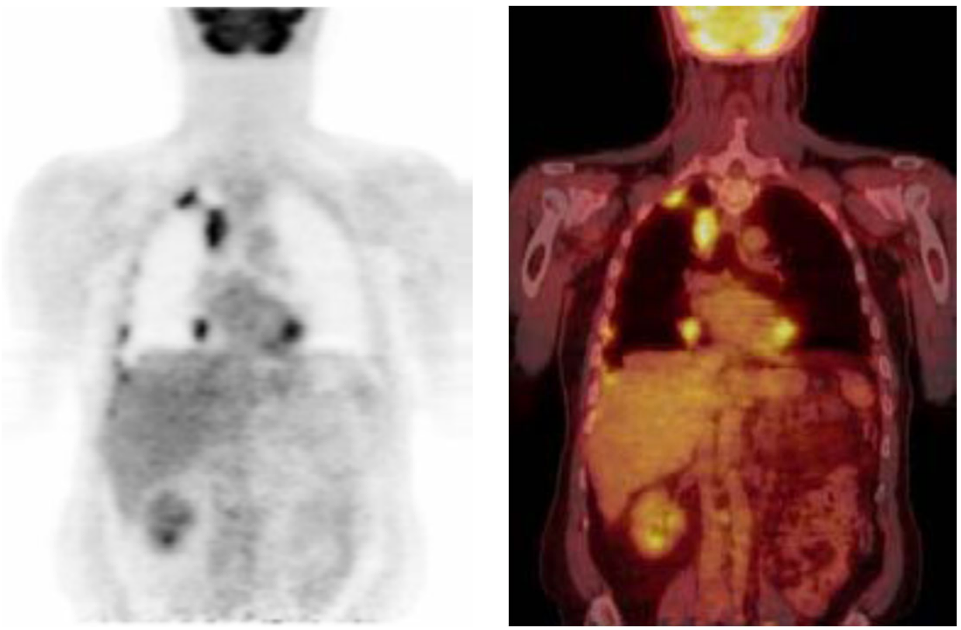
Chapter 2: Mathematical Preliminaries

Image Quality

What makes for a Good Quality Image?

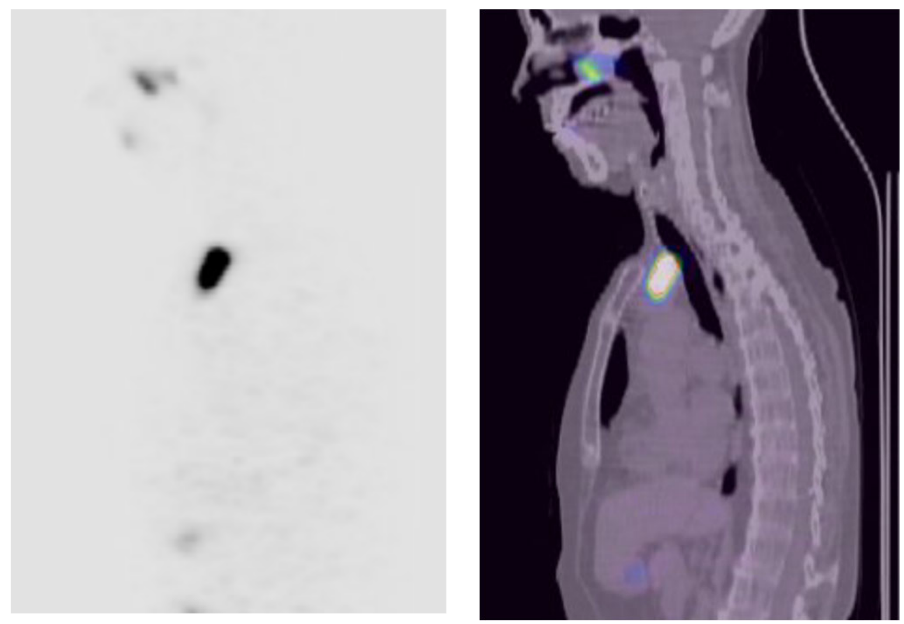
Sharpness, Contrast, Low noise …

Clinical Applications of PET

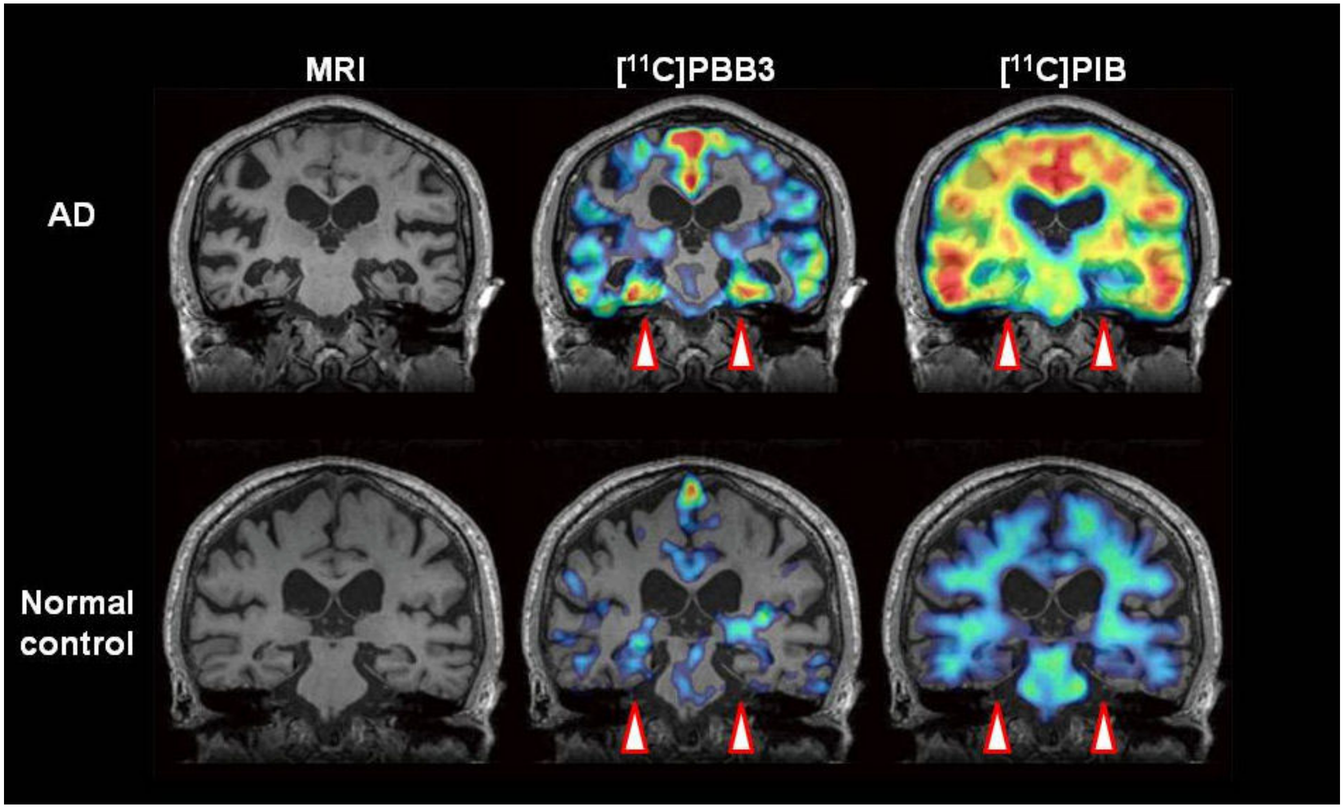


FDG study: lung or ribs?

I-124 study: bone or soft tissue?



Positron Decay and PET Imaging

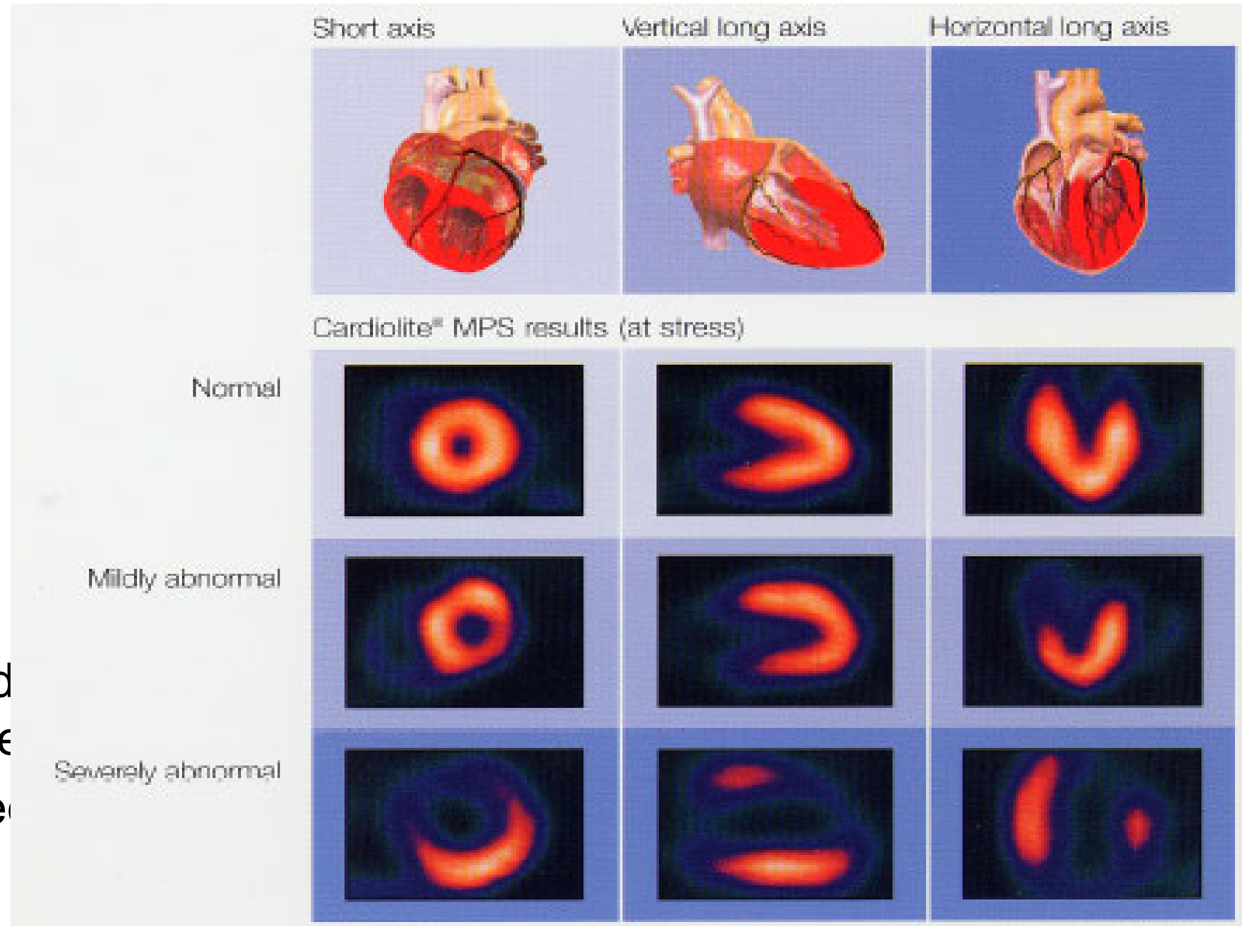


Negative and positive [C-11]PiB PET images. [C-11]PiB PET images taken in the axial plane at levels indicated in the sagittal MRI image at the far left. The scans in the top row illustrate the 79-year-old [C-11]PiB(−) DLB subject that is the focus of the current study. There is no evidence of [C-11]PiB retention except for nonspecific retention in the white matter. The scans in the bottom row illustrate a 65-year-old [C-11]PiB(+) AD subject showing high [C-11]PiB retention throughout the neocortex and the striatum

What do MPI images look like?

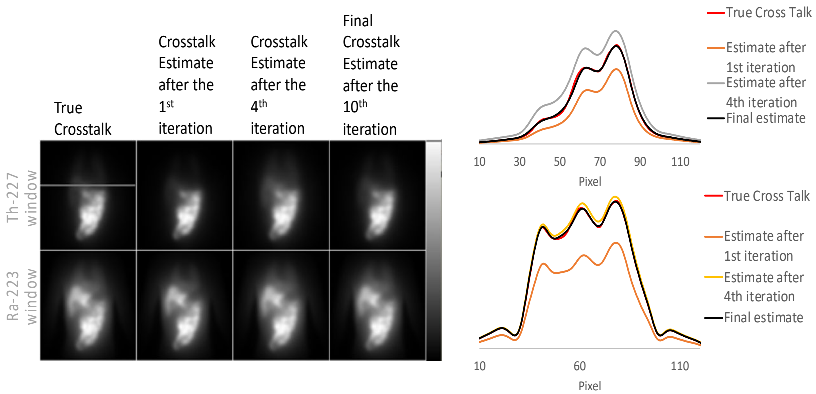
In a typical nuclear cardiac imaging exam, the physician reviews:

- Static "Summed Perfusion Image
- Dynamic "Gate Images"

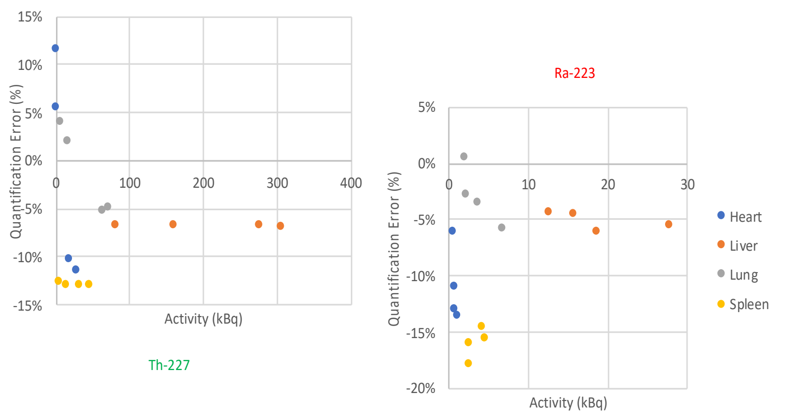


Perfusion Images are viewed in three orientations:
SA – Short Axis
VLA – Vertical Long Axis
HLA - Horizontal Long Axis

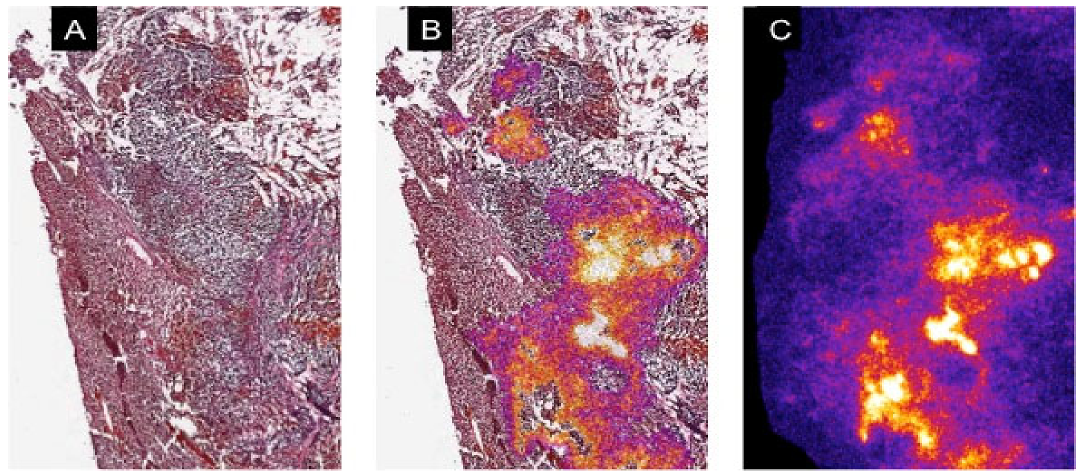
Hyperspectral SPECT Imaging of TAT



Estimated crosstalk of Th227/Ra223 imaging.



VOI quantification. Left: Th-227; Right: Ra-223.



H&E staining of VX2 tumor and surrounding liver section (A) and corresponding α-Camera image (C) 3 hours post-injection of ^{225}Ac-DOTAGA-TDA. (B) Merged image showing segmented high-intensity regions overlayed on H&E stained slide. Segmented regions do not exactly match the tumor boundaries due to differences in tissue processing and because the comparison is made across two different 12-μm-thick sections.

Data presented in this slide is kindly provided by our collaborators, Dr. Eric Frey and Dr. Yong Du at Johns Hopkins Medical School

Purpose of the Lecture

- Applies to all types of images
- 'Quality': subjective notion, dependent on image function
- Bottom line outcome measure of a radiological image is its usefulness in determining an accurate diagnosis
- Understanding the image characteristics that comprise image quality is important so that radiologists can recognize problems and articulate their cause
- Introduction to the terminology used for various metrics by physicists and engineers to measure image quality, e.g., contrast, spatial resolution and noise

Covered in lecture

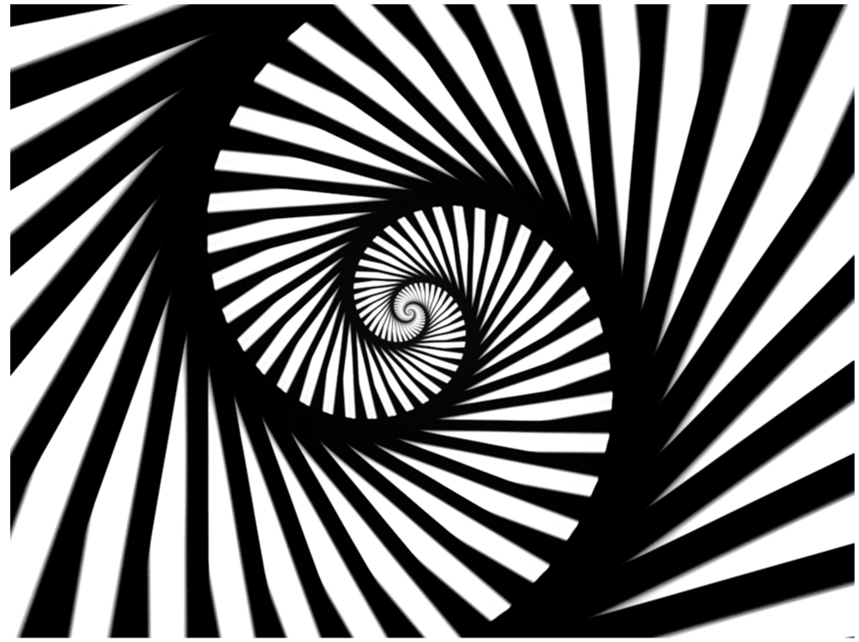
A Revisit to Key Image Quality Measures

Contrast

- What is contrast of an image?
  How to quantify contrast?
  How an imaging system affect contrast?

Covered in lecture

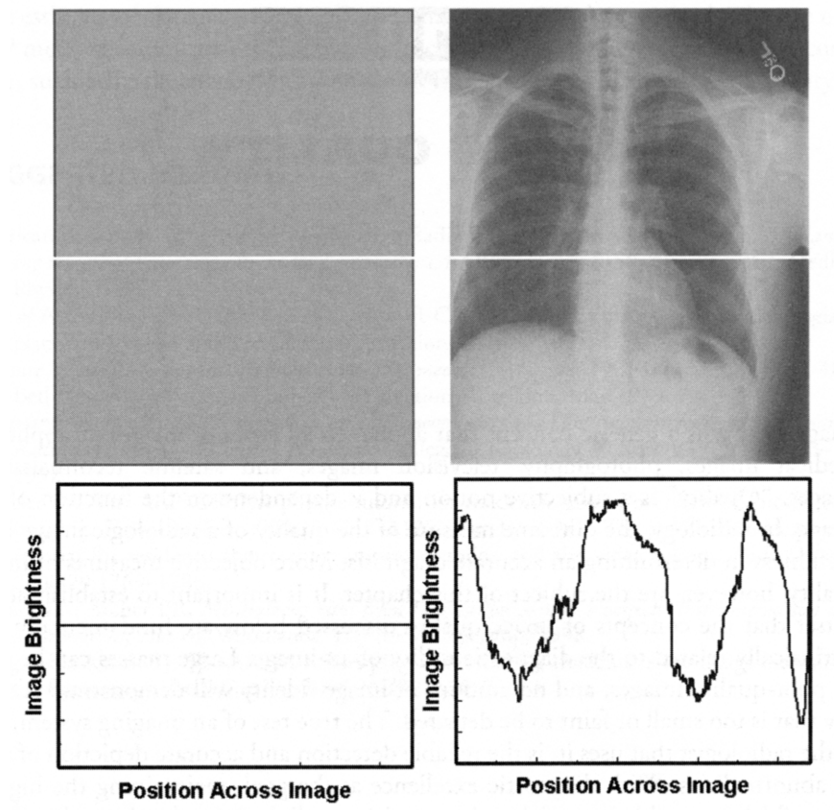
Contrast



Covered in lecture

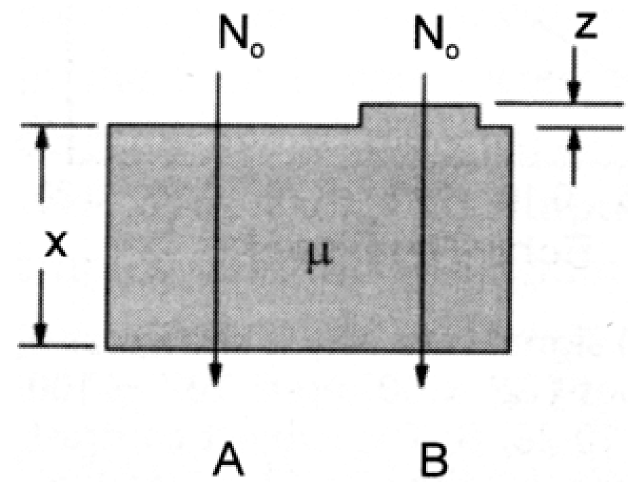
Contrast

- What is contrast?
- The difference in the image gray scale between closely adjacent regions of the image.
- Medical image contrast is the result of many steps during image acquisition, processing and display

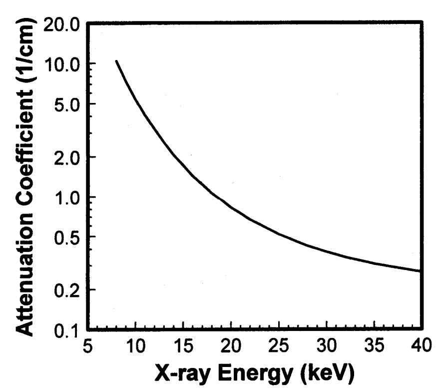


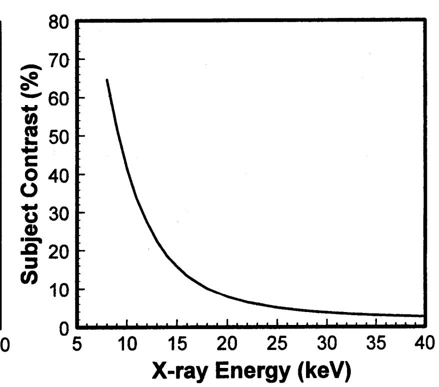
Covered in lecture

Subject (or Object) Contrast



- Difference in some aspects of the signal prior to it being recorded
- Consequence of fundamental differences in the object, e.g., in x-ray intensity based on attenuation
- $C = (A\text{-}B)/A$

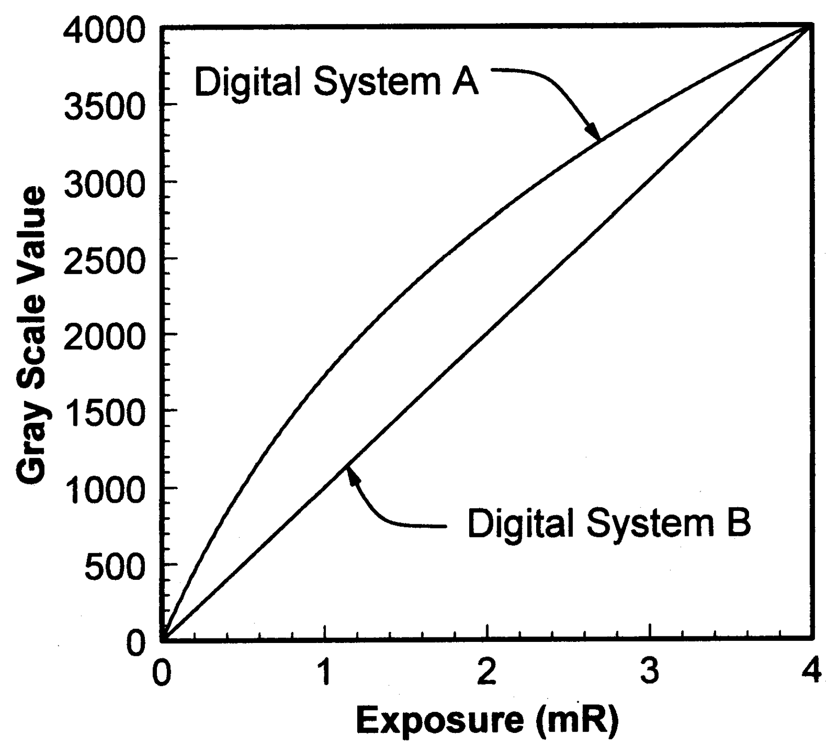




Covered in lecture

Detector Contrast

- Detector Contrast (C_d)
- A detector's characteristics play an important role in producing contrast in the final image
- C_d determined principally by how the detector 'maps' detected energy into the output signal
- Characteristic curve: input radiation exposure to output value (analog or digital)



Covered in lecture

A Revisit to Key Image Quality Measures

Modulation

- What is modulation?
- Definition of modulation transfer function
- How to experimentally measure MTF

Covered in lecture

Modulation

Recall that an arbitrary signal can be decomposed into the weighted sum of periodic signals, such as sinusoidal waves, so study the response of an imaging system to such signals provide a effective approach for quantifying the image quality.

Covered in lecture

Modulation

- The modulation m_f is an effective way to quantify the contrast of a periodic signal

$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}}\,.$$

- In general, m_f is refer to as the contrast of a periodic signal f(x,y) relative to its average value.

- So within two signals, f(x,y) and g(x,y), with the same average value , f(x,y) is said to have more contrast if $m_f > m_g$.

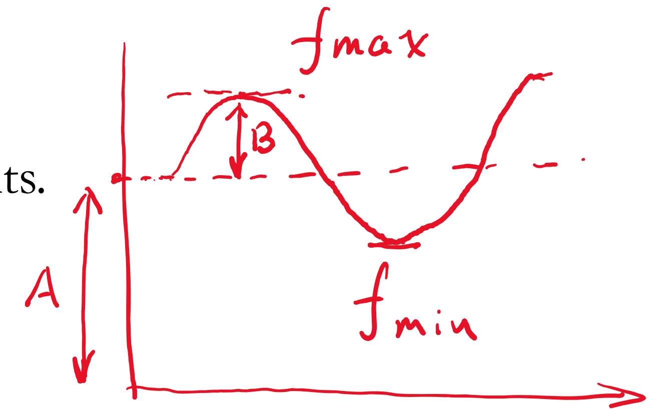
Covered in lecture

Modulation

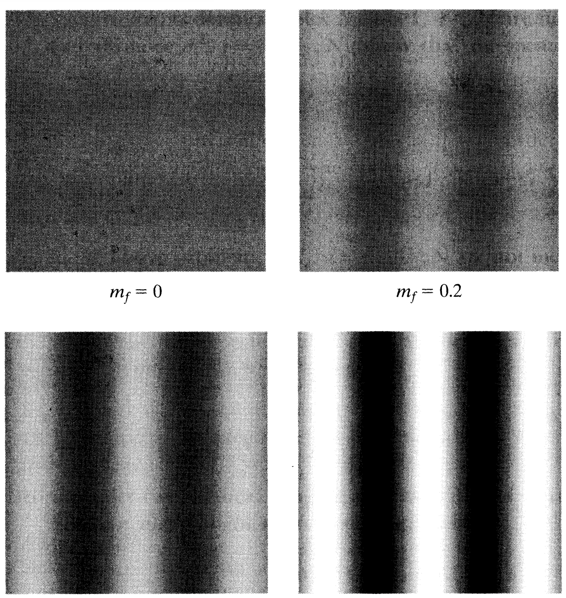
- Suppose an input signal function

$$f(x, y) = A + B\sin(2\pi u_0 x)\,,$$

where A>B and both are non-negative constants.



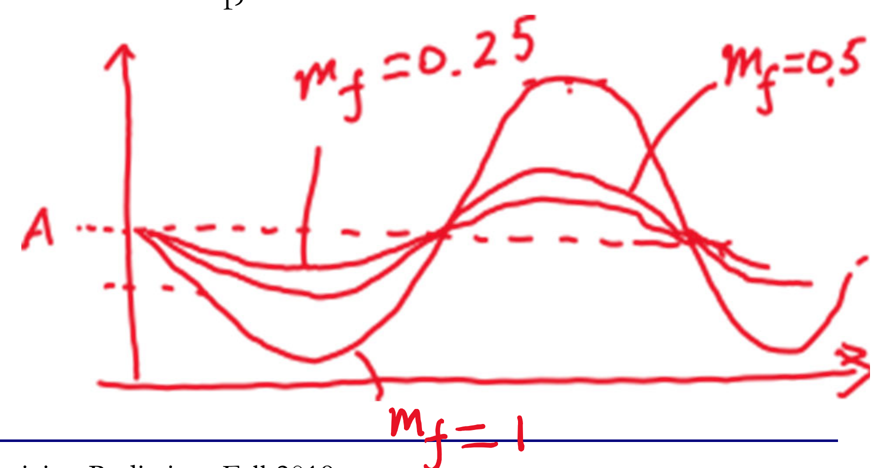
$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}}\,. \quad \Rightarrow \quad m_f = \frac{B}{A}\,.$$



$m_f = 0$ $m_f = 0.2$

$m_f = 0.5$ $m_f = 1$

Greater m_f, more contrast



Modulation

- Now let this signal to pass through an LSI imaging system. Suppose an input signal function. Since

$$f(x,y) = A + B\sin(2\pi u_0 x) = A + \frac{B}{2j}\left[e^{j2\pi u_0 x} - e^{-j2\pi u_0 x}\right],$$

- Suppose the system impulse response function h(x,y) is circularly symmetric, then

$$g(x,y) = AH(0,0) + B\,|H(u_0,0)|\sin(2\pi u_0 x).$$

- Then

  $g_{min} = AH(0,0) - B|H(u_0,0)|$ and $g_{\max} = AH(0,0) + B|H(u_0,0)|$,

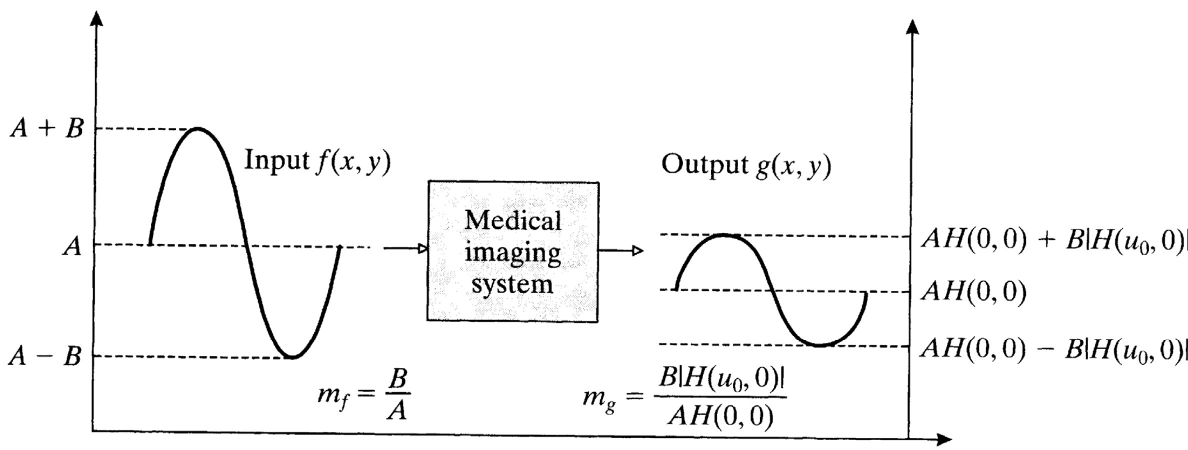
- The modulation of the input and output signals are

$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}} = \frac{B}{A}, \quad \text{and} \quad m_g = \frac{g_{\max} - g_{\min}}{g_{\max} + g_{\min}} = \frac{B|H(u_0,0)|}{AH(0,0)} = m_f \frac{|H(u_0,0)|}{H(0,0)}$$

Covered in lecture

Modulation

- The effect of an LSI having circular symmetric impulse response function on an input sinusoidal signal is to scale the input signal by a factor equal to the magnitude spectrum at the same frequency u_0.



- It is often true that $H(0,0) \cong 1$, and $H(U_0,0) < 1$. So that the output signal has less contrast, $m_g < m_f$.

Covered in lecture

Modulation

- Now let this signal to pass through an LSI imaging system. Suppose an input signal function. Since

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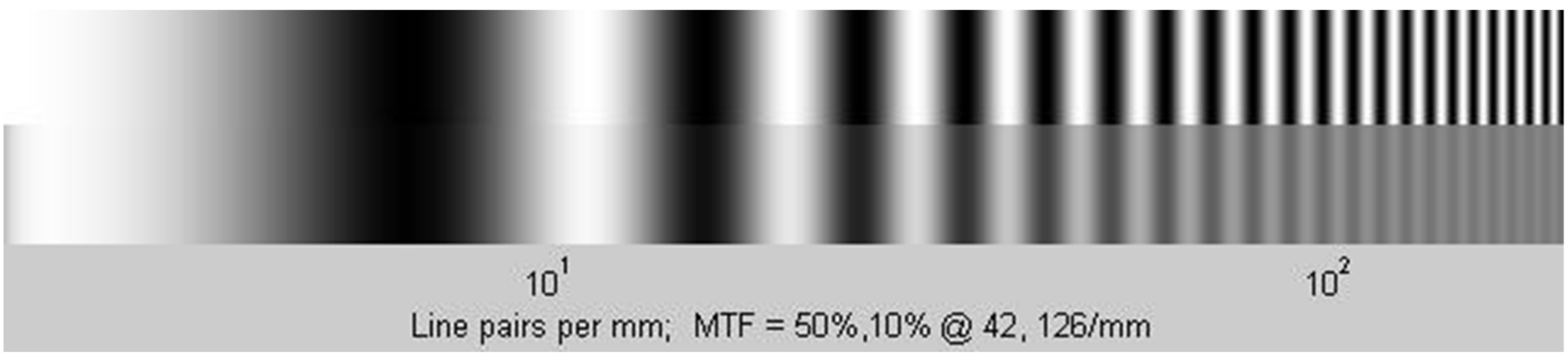
- Suppose the system impulse response function h(x,y) is circularly symmetric,

$H(u,v) = \mathcal{F}[h(x,y)]$

$$g(x, y) = AH(0, 0) + B\,|H(u_0, 0)|\sin(2\pi u_0 x)\,.$$



Covered in lecture

Modulation

- Now let this signal to pass through an LSI imaging system. Suppose an input signal function. Since

$$f(x,y) = A + B\sin(2\pi u_0 x) = A + \frac{B}{2j}\left[e^{j2\pi u_0 x} - e^{-j2\pi u_0 x}\right],$$

- Suppose the system impulse response function h(x,y) is circularly symmetric,

$$g(x,y) = AH(0,0) + B\,|H(u_0,0)|\sin(2\pi u_0 x).$$

- So the modulation of the output signal is

$$m_g = \frac{B|H(u_0,0)|}{AH(0,0)} = m_f\frac{|H(u,0)|}{H(0,0)}.$$

- Modulation Transfer Function (MTF).

$$\mathrm{MTF}(u) = \frac{m_g}{m_f} = \frac{|H(u,0)|}{H(0,0)}.$$

Covered in lecture

Modulation Transfer Function (MTF)

- In order to fully quantify the response of an LSI for an arbitrary signal, f(x,y), we would need to know the response of the system to sinusoidal signals at different frequencies.

- Modulation Transfer Function (MTF).

$$\mathrm{MTF}(u) = \frac{m_g}{m_f} = \frac{|H(u,0)|}{H(0,0)} .$$

- MFT is, in effect, the "frequency response function" of a given imaging system. It is normally evaluated for positive frequencies only.

- Most imaging systems lead to decreased contrast, so that

$$0 \leq \mathrm{MTF}(u) \leq \mathrm{MTF}(0) = 1, \qquad \text{for every } u ,$$

Covered in lecture

Modulation Transfer Function (MTF)

- In case of non-isotropic impulse response function (h(x,y) is not circularly symmetric), the MTF can be defined as

$$\mathbf{MTF}(u,v) = \frac{m_g}{m_f} = \frac{|H(u,v)|}{H(0,0)},$$

- A typical MTF of an imaging system

$$|H(u,v)| = \sqrt{H_R^2(u,v) + H_I^2(u,v)}$$

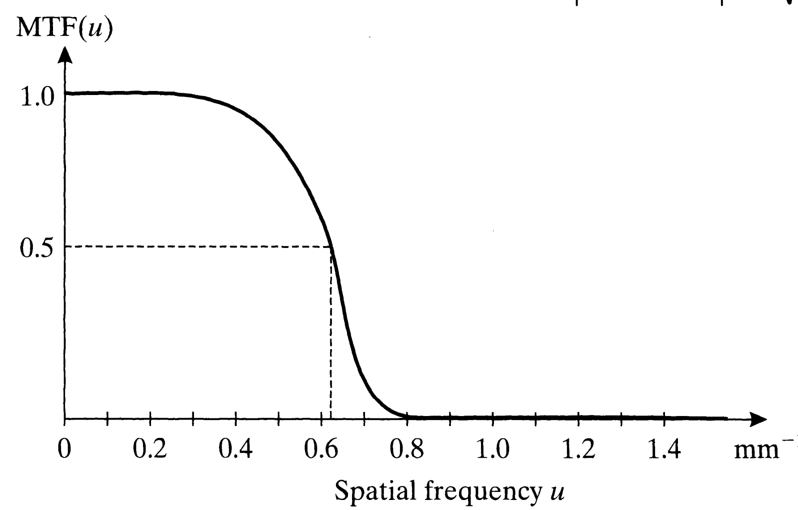


Figure 3.3
A typical MTF of a medical imaging system.

Modulation Transfer Function (MTF)

- In case of non-isotropic impulse response function (h(x,y) is not circularly symmetric), the MTF can be defined as

$$\mathrm{MTF}(u,v) = \frac{m_g}{m_f} = \frac{|H(u,v)|}{H(0,0)},$$

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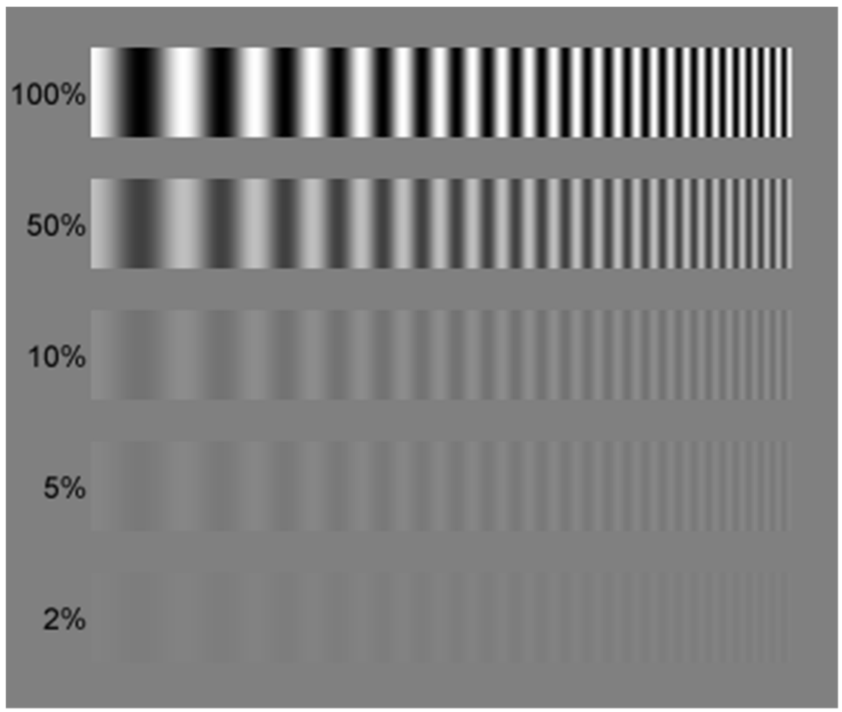


Uniform MTF

Covered in lecture

Modulation Transfer Function (MTF)

$$MTF(u) = \frac{m_g}{m_f} = \frac{|H(u,0)|}{H(0,0)}$$

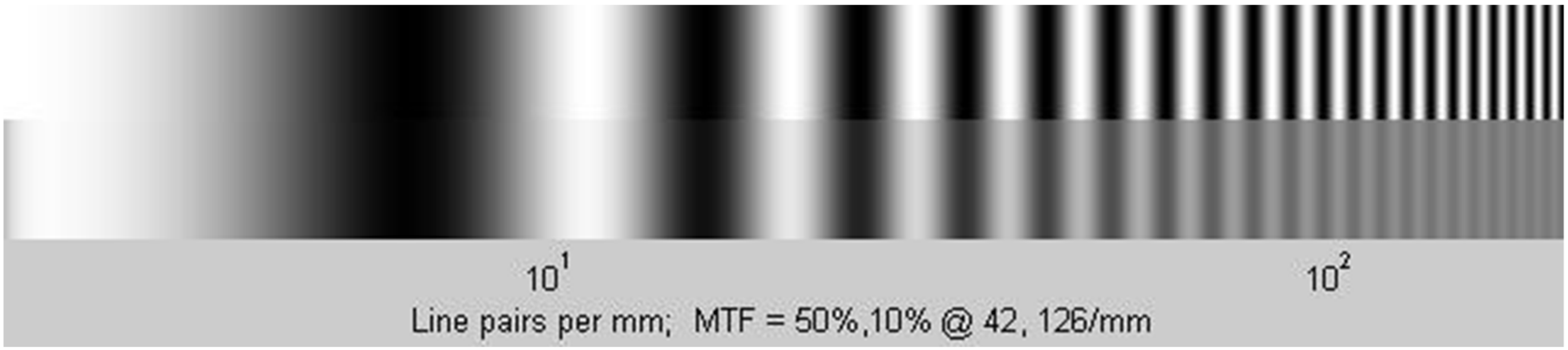
If an imaging system is designed to faithfully reproduce a DC signal, then ((0,0)=1. So the MTF becomes

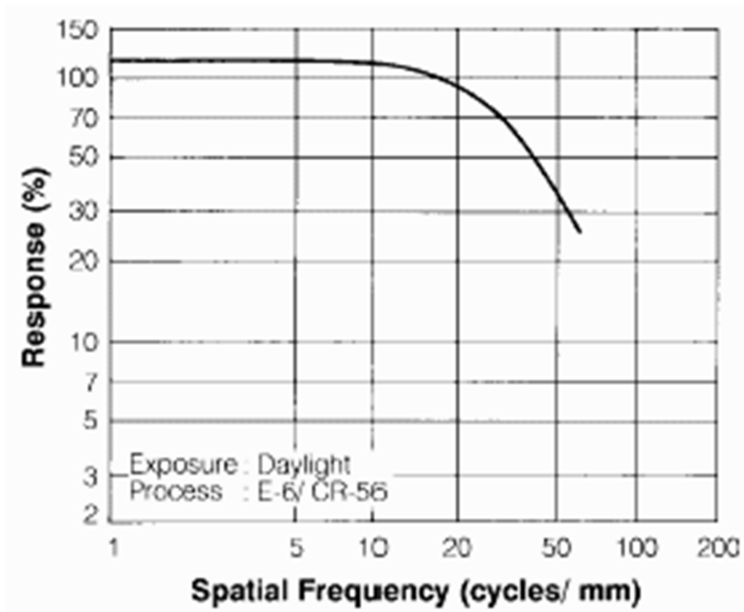
$$MTF(u) = \frac{m_g}{m_f} = |H(u,0)| \ .$$

Furthermore, if since the response of the system is assumed to be isotropic, then fourier transform of the system response function

Modulation Transfer Function (MTF)

$$\mathrm{MTF}(u) = \frac{m_g}{m_f} = \frac{|H(u,0)|}{H(0,0)}.$$





System Cascade

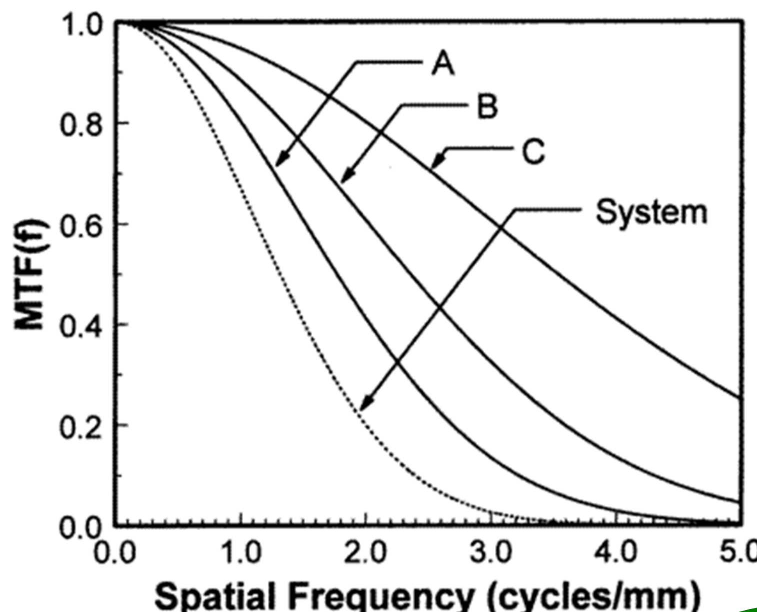
- Modulation Transfer Function (MTF).

$$g(x, y) = h_K(x, y) * \cdots * (h_2(x, y) * (h_1(x, y) * f(x, y)))\,.$$

- The overall MTF is the product of the MTF for sub-systems:

$$MTF_{total}(u) = \prod_{i=1}^{i=I} MTF_i(u)$$

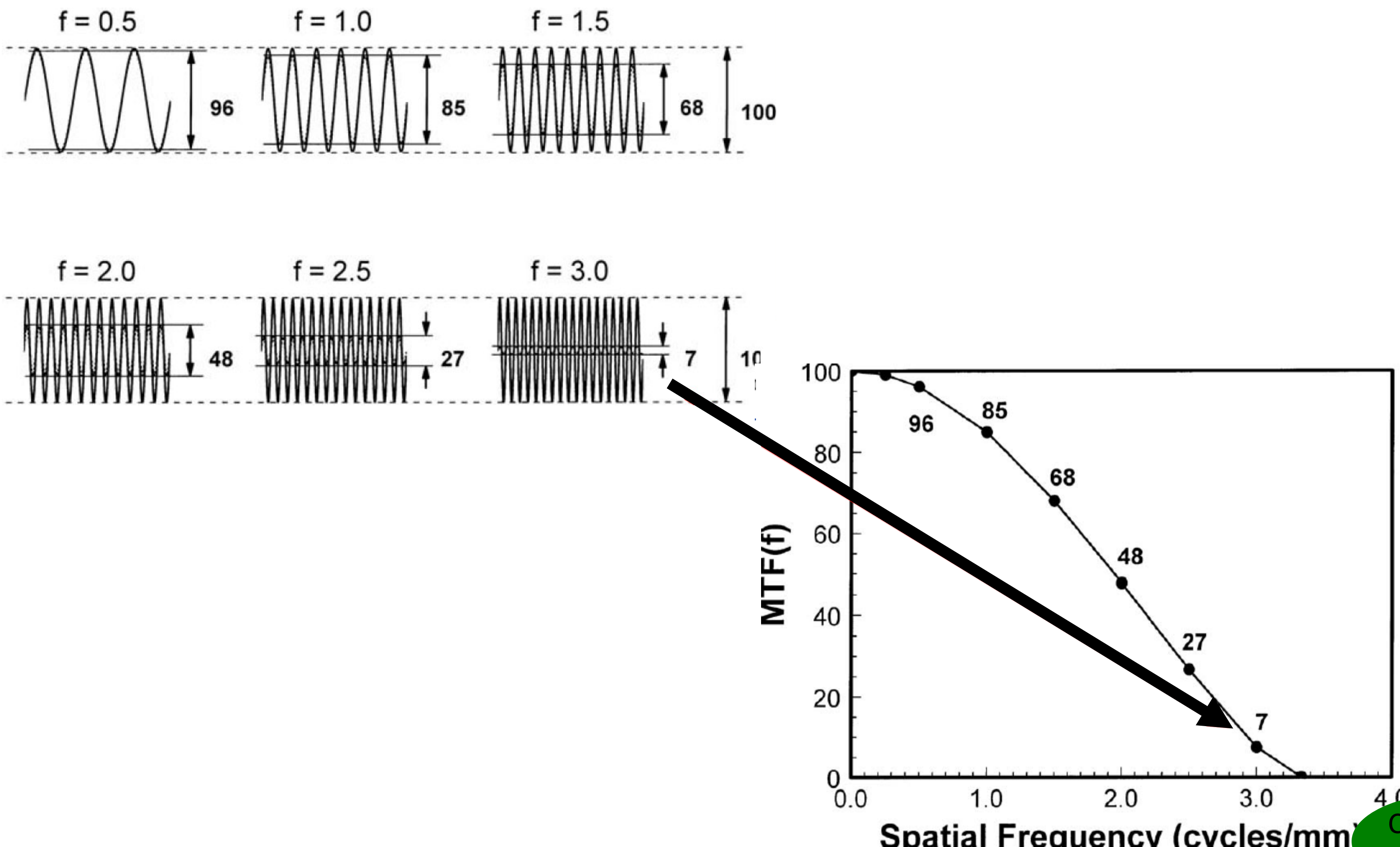
- MTF of an imaging system that can be modeled as a chain if systems is often determined by the MTF of the worst system in the cascade.



Covered in lecture

Modulation transfer Function -- Revisited

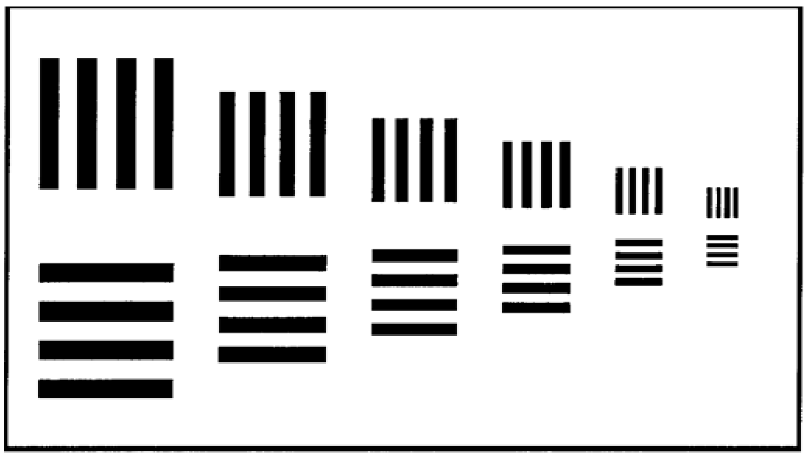
- Modulation Transfer Function (MTF).

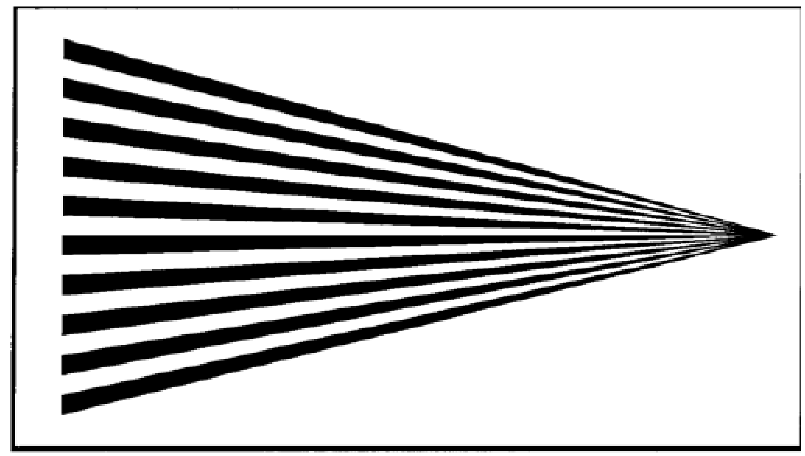


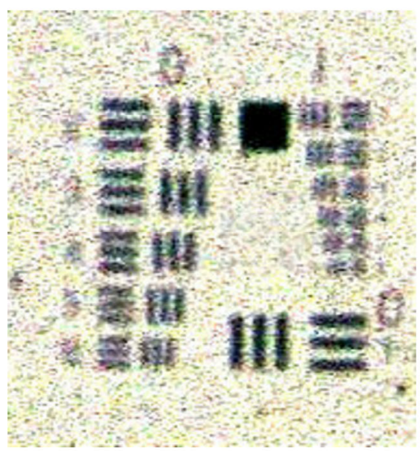
Covered in lecture

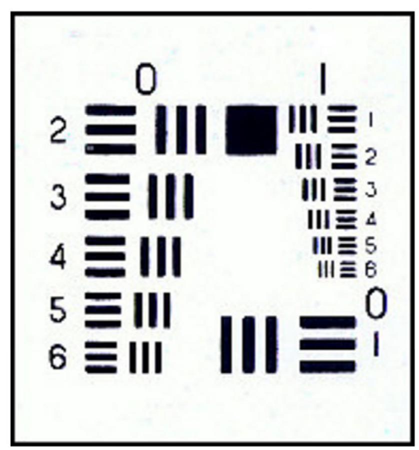
Modulation Transfer Function -- Revisited

Standard Test Chart for Determining the Modulation Transfer Function (MTF) of an Imaging System.



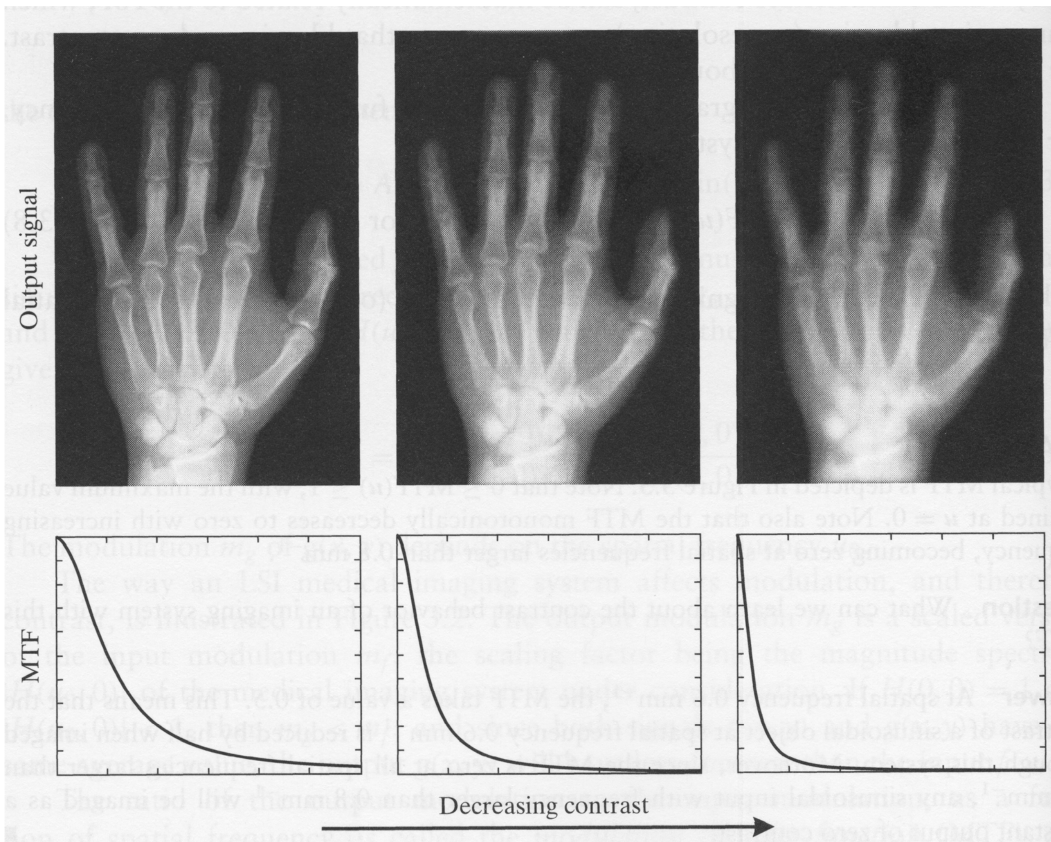






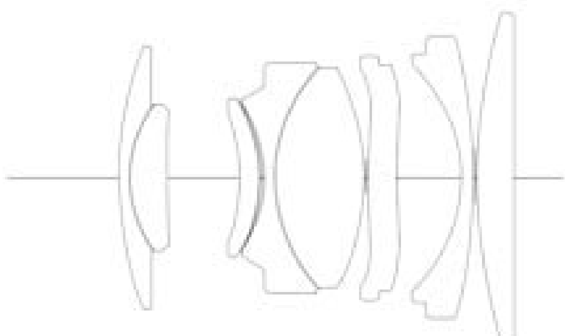
Modulation Transfer Function (MTF)

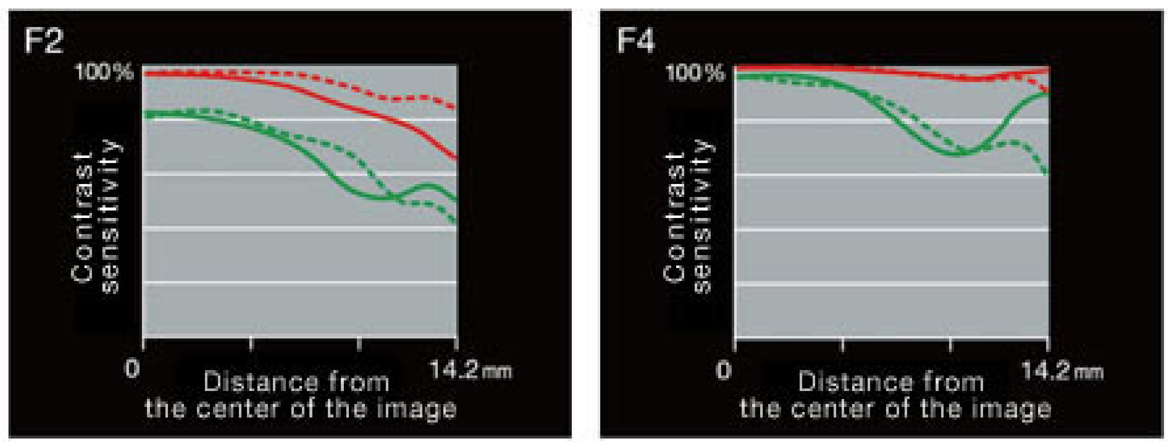
- An example of the same signal passing through three imaging systems with decreasingly poor MTF, which leads to decreasing contrast in images



Covered in lecture

Modulation Transfer Function (MTF)



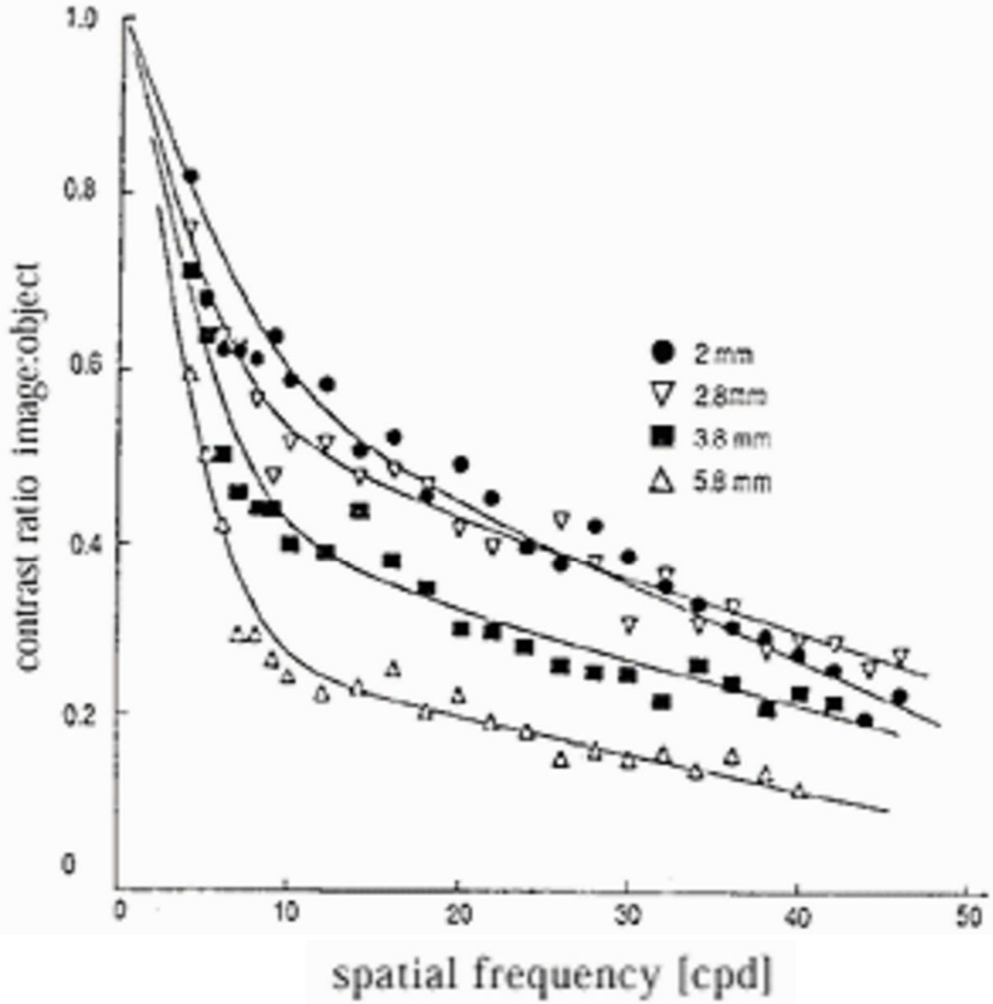


	S	M
10 lp/mm	—— (red)	------ (red)
30 lp/mm	—— (green)	------ (green)

S : Sagittal M : Meridional

Optical modulation transfer function (MTF) of the human eye

- MTF is measured directly with sinewave gratings.
- The optical modulation transfer function (MTF) can be interpreted as Fourier transform of the optical LSF.



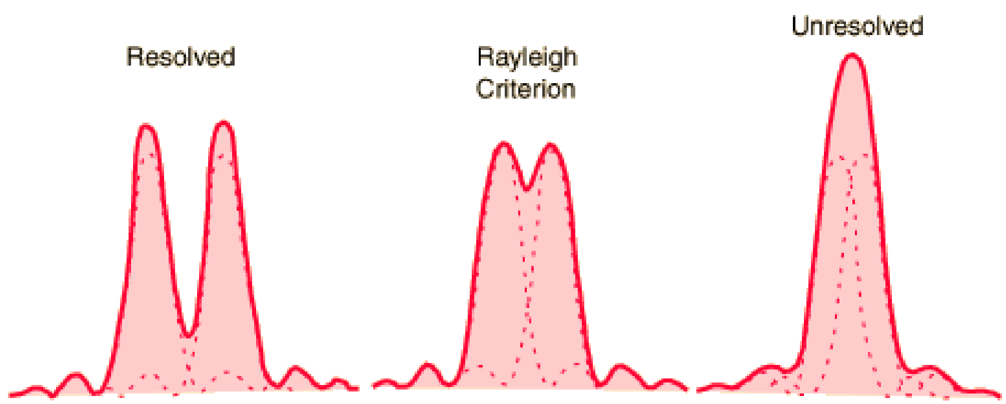
A Revisit to Key Image Quality Measures

Spatial Resolution

- How to quantify spatial resolution?
- How to measure spatial resolution?
- The relationship between spatial resolution and modulation transfer function (MTF)?

Spatial Resolution

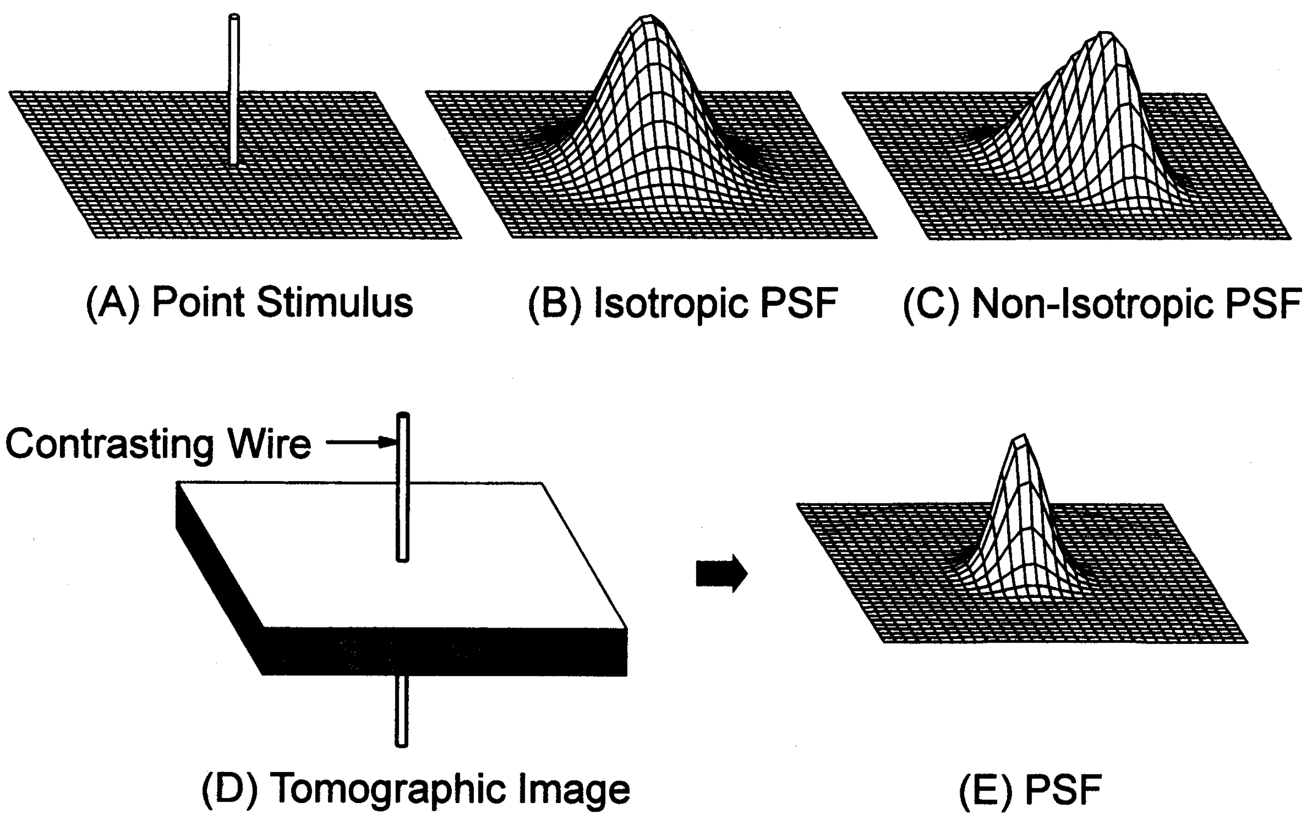
- Resolution: the ability of an given imaging system to accurately depict two distinct events in space, time or frequency respectively.
- Resolution can also be thought as the degree of blurring, smearing.



- Spatial resolution is fully described by the point-spread function or impulse response function, h(x,y).

The Point Spread Function (PSF)

- One method is to stimulate the imaging system with a point impulse and observe the resulting point-response function.



Full-width at Half Maximum (FWHM) of The Point Spread Function (PSF)

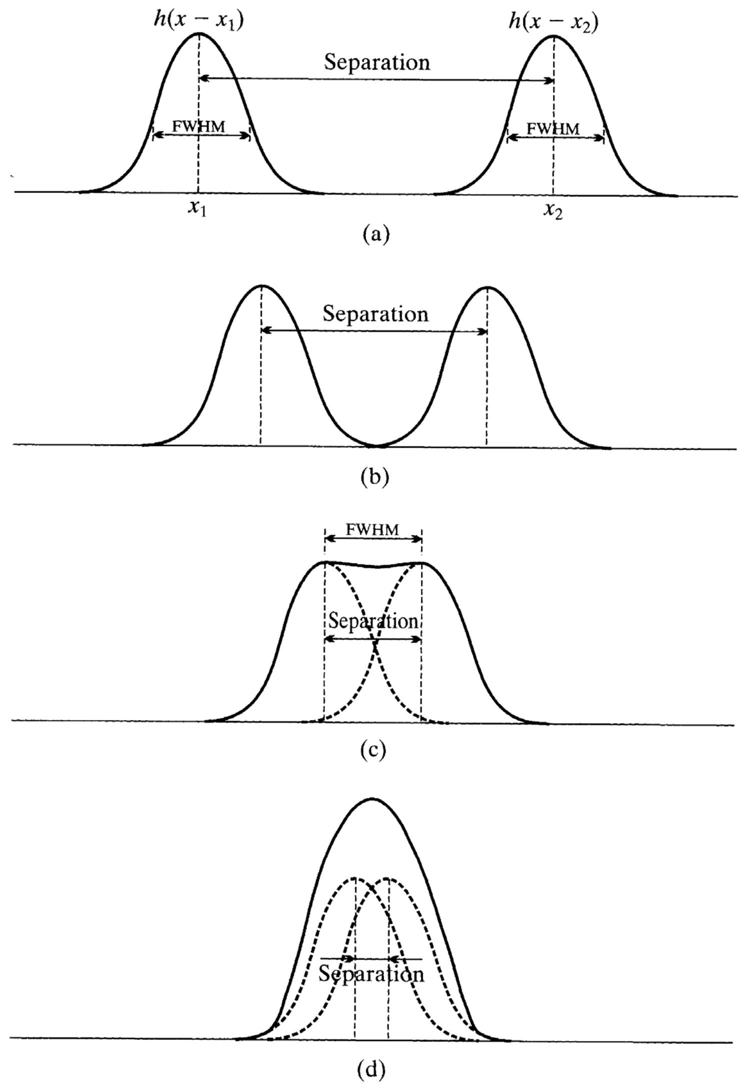
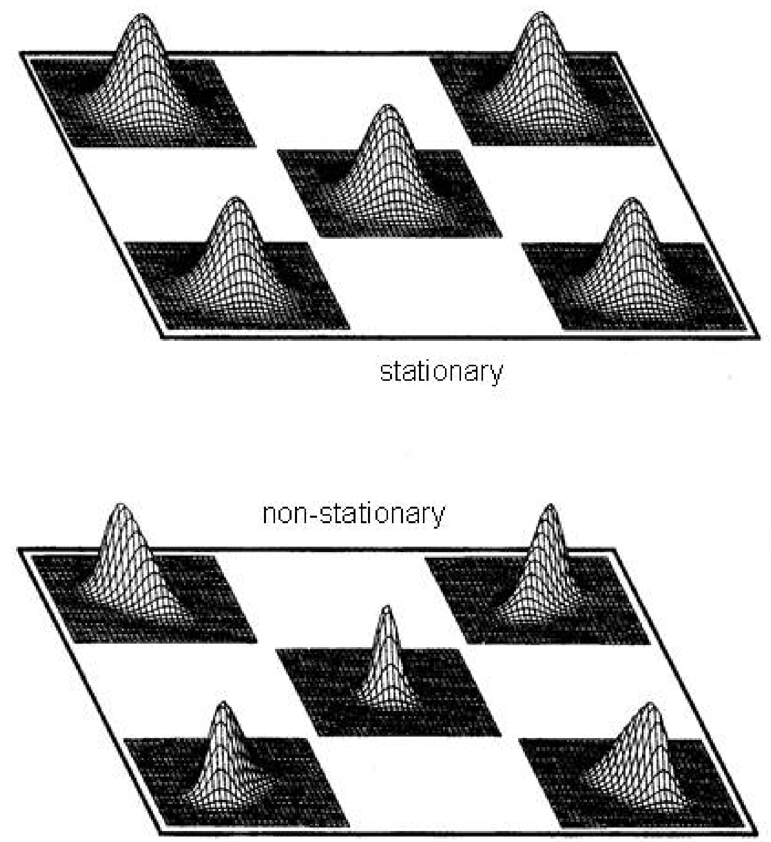


Figure 3.6
An example of the effect of system resolution on the ability to differentiate two points. The FWHM equals the minimum distance that the two points must be separated in order to be distinguishable.

Stationary and Non-stationary PSF

- Spatial variation of the PSF is another important aspect of an given imaging system.



Other Ways to Measure the Spatial Resolution
Line Response Function

- The resolution of an imaging system can also be estimated with the line-spread function
- Given a line impulse

$$\textit{line impulse } f(x,y) = \delta_\ell(x,y) = \delta(x\cos\theta + y\sin\theta - \ell)$$

- Assuming the impulse response function of the imaging system h(x,y) is isotropic, it is sufficient to consider the response to a vertical line through the origin. For this case, line response function is

$$\begin{aligned} g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta) f(x-\xi, y-\eta)\, d\xi\, d\eta\,, \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\xi,\eta)\delta(x-\xi)\, d\xi\right] d\eta\,, \\ &= \int_{-\infty}^{\infty} h(x,\eta)\, d\eta\,, \end{aligned}$$

$$\boxed{\textit{Line Response Function}:\ l(x) \equiv \int_{-\infty}^{\infty} h(x,\eta)d\eta}$$

Line Response Function

$$\textit{Line Response Function}: l(x) \equiv \int_{-\infty}^{\infty} h(x,\eta)d\eta$$

- The 1-D Fourier transform of the line spread function is

$$\begin{aligned} L(u) &= \mathcal{F}_{1D}[l](u)\,, \\ &= \int_{-\infty}^{\infty} l(x)e^{-j2\pi ux}\,dx\;, \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,\eta)e^{-j2\pi ux}\,dx\,d\eta\,, \\ &= H(u,0). \end{aligned}$$

Remember that

$$MTF(u) = \frac{m_g}{m_f} = |H(u,0)|$$

- So the values of the Fourier transform of the LSF crossing a horizontal line passing through the origin is sufficient for describing the LSF

LSF and MTF

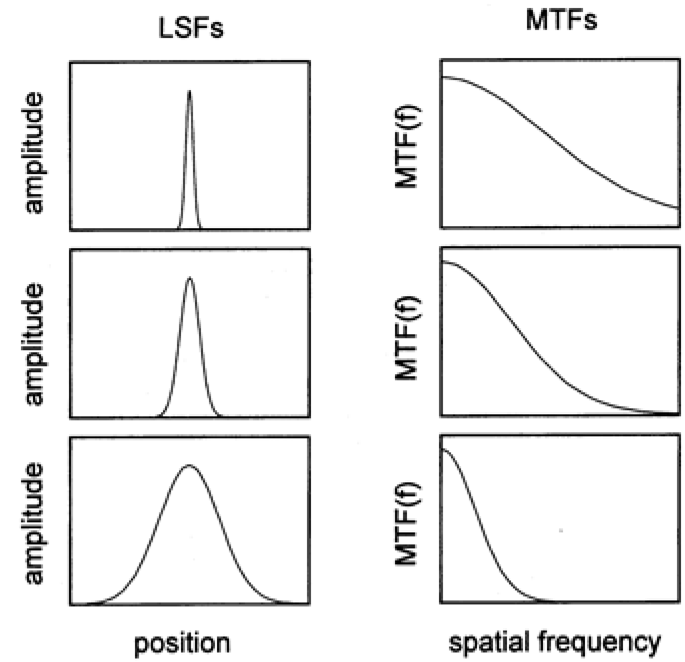
- Modulation Transfer Function (MTF).

$$\mathrm{MTF}(u) = \frac{m_g}{m_f} = \frac{|H(u,0)|}{H(0,0)} = \frac{|L(u)|}{L(0)}, \qquad \text{for every } u.$$

- For a “reasonable” imaging system, the L(0)=1, so that

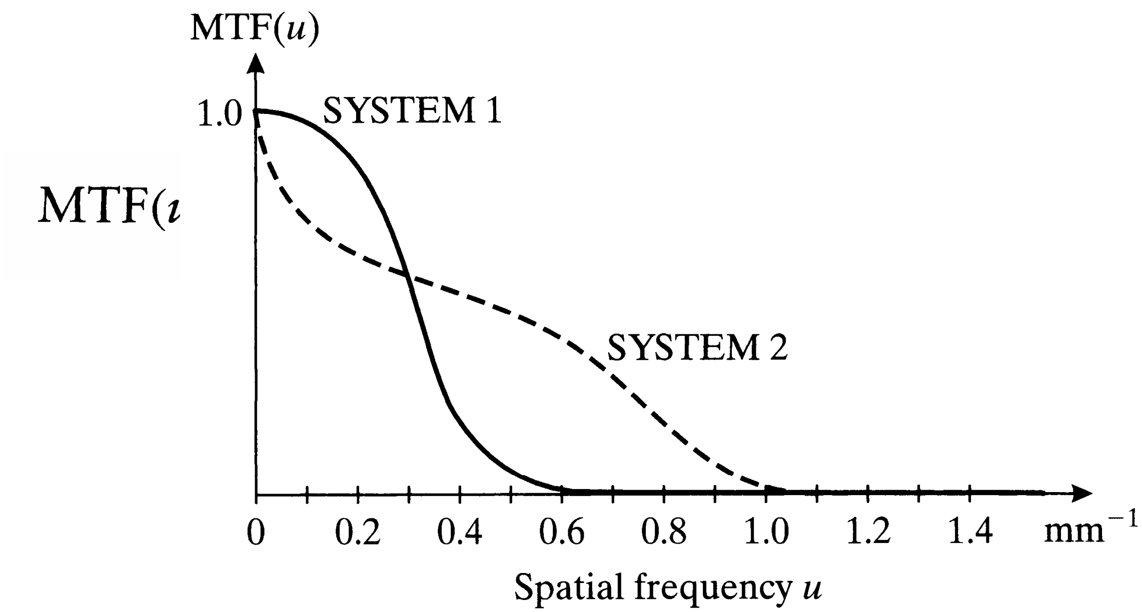
$$\mathrm{MTF}(u) = L(u)$$

- MTF is an effective way to compare two imaging systems in terms of spatial resolution and contrast.



$$\begin{aligned} g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta) f(x-\xi, y-\eta)\, d\xi\, d\eta\,, \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\xi,\eta)\delta(x-\xi)\, d\xi\right] d\eta\,, \\ &= \int_{-\infty}^{\infty} h(x,\eta)\, d\eta\,, \quad \equiv l(x), \qquad L(u) = \mathcal{F}_{1D}[l(x)] \end{aligned}$$

LSF and MTF

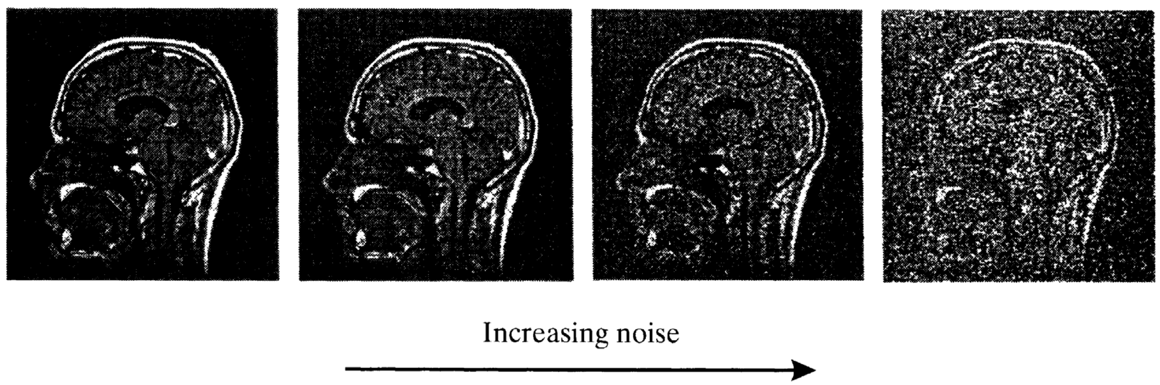


- System 1 has better low frequency contrast and it is better for imaging coarse features.
- System has a better high frequency contrast. It is better for resolving fine details.

- The resolution of a system is related to the higher frequency components and the cut-off frequency of the MFT.

Image Noise

- The random fluctuation is referred to as the image noise. That has dramatic effects on the subsequent analysis, for example, for signal detection and quantification tasks …



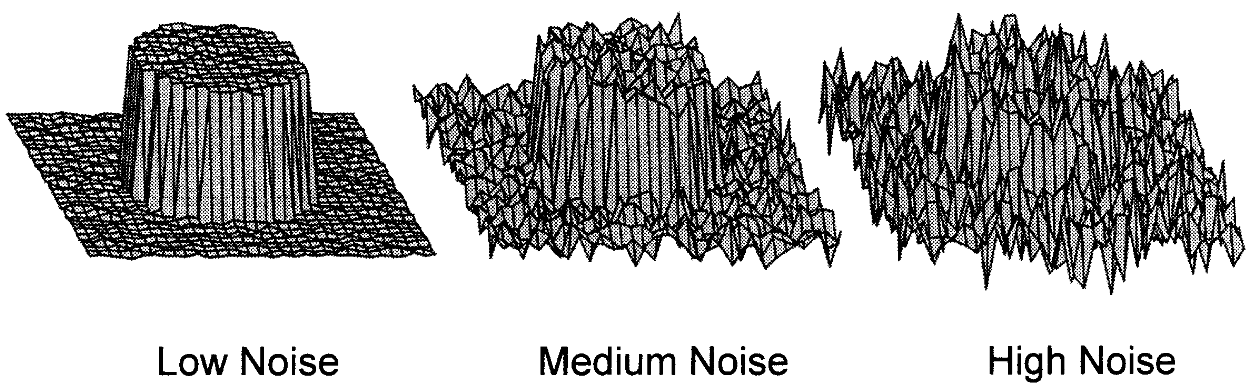
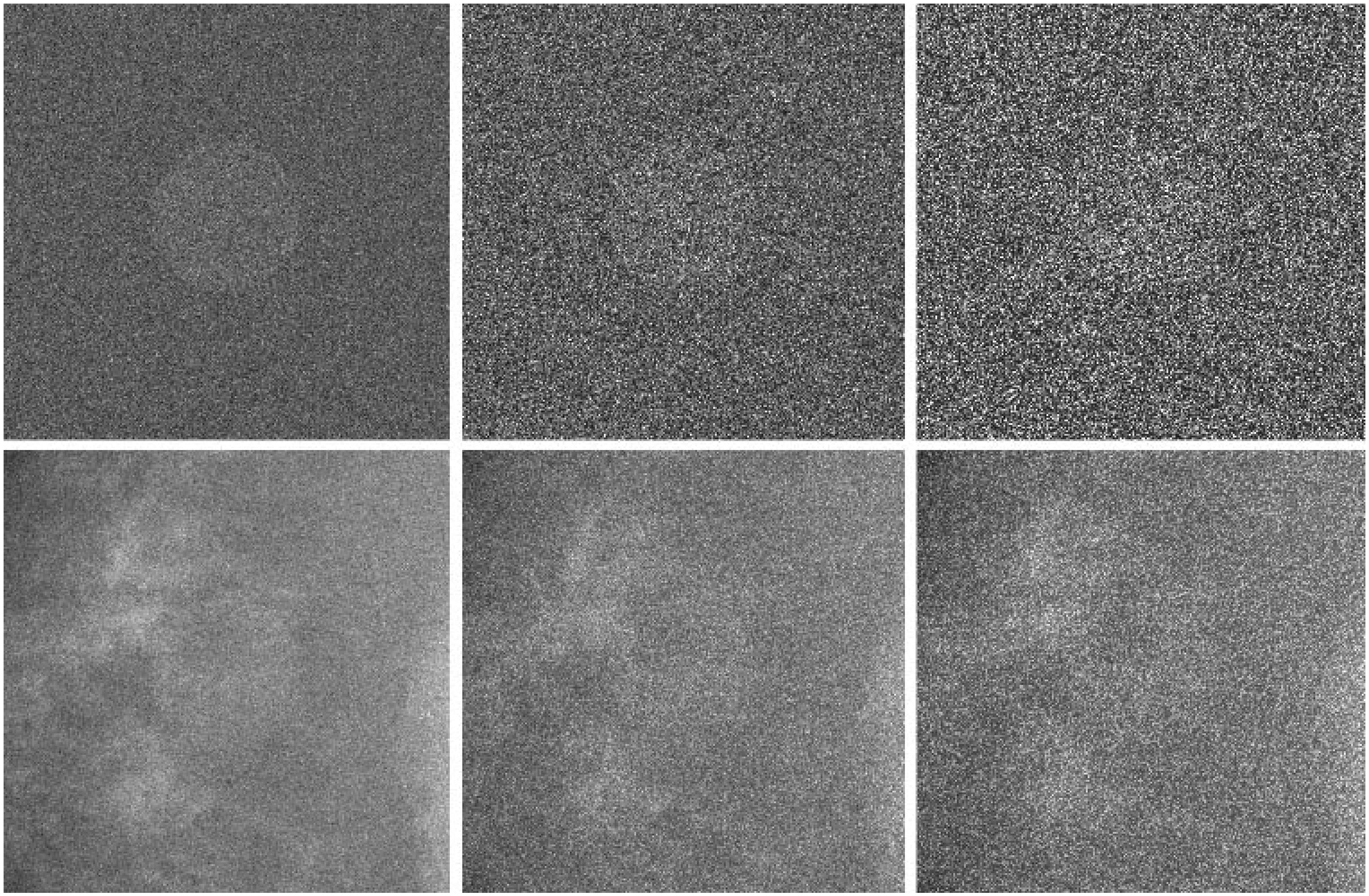


Image Noise Reduces Contrast



Where is the noise coming from?

Image Noise

- Due to the random nature of the data acquired by any imaging system, the output images are normally multivariate random variables.

Input signal ⇨ Data acquisition hardware ⇨ Noisy Data ⇨ Data processing ⇨ Final image

noise propagation

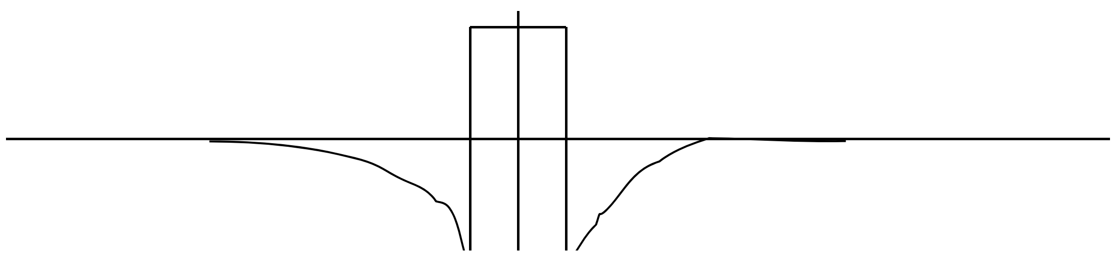
An Example – Noise on Images Acquired with FBP

The Ram-Lak filter in spatial domain

$$h_{\mathrm{RL}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{\mathrm{RL}}(\omega)\exp(ix\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} |\omega| \exp(ix\omega)\, d\omega$$

$$= 2B^2 \operatorname{sinc}(2\pi Bx) - B^2 \operatorname{sinc}^2(\pi Bx)$$



$$\hat{f}(x, y) = \frac{1}{\pi}\int_0^{\pi} d\phi \int_{-\infty}^{\infty} dx'\, p_\phi(x')h(x\cos\phi + y\sin\phi - x')$$

$$\boxed{\hat{f}_i = \sum_{j=1}^{M} S_{ij} \cdot g_j}$$

Filtered Back-projection

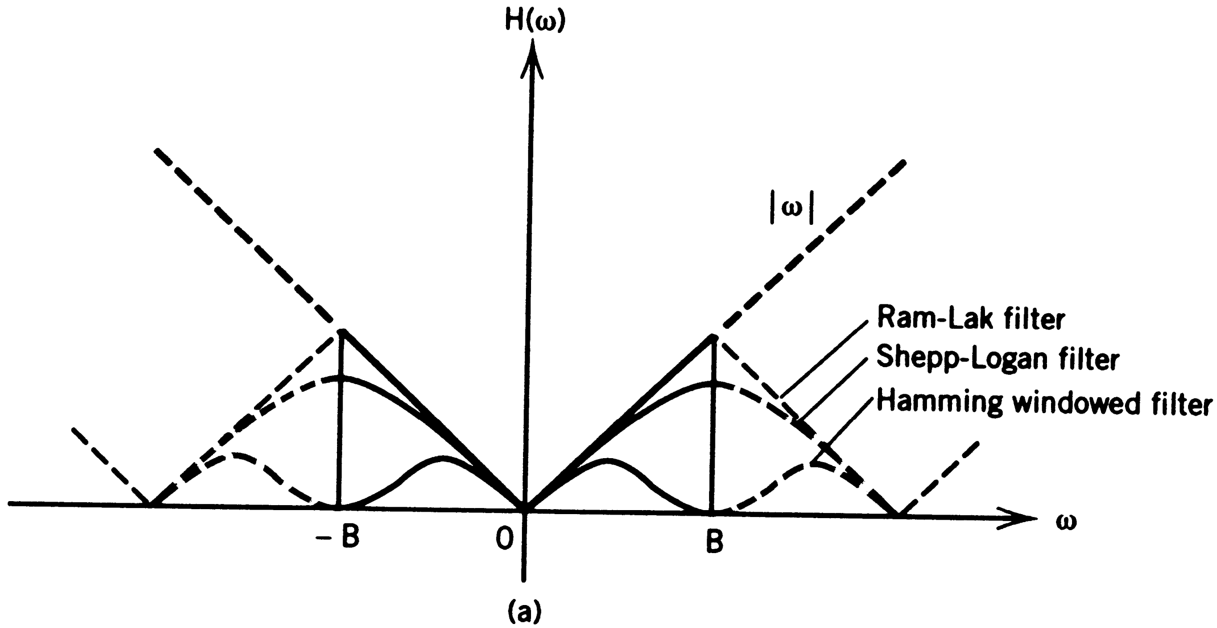


Figure 3-4 (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter.

Inverse Radon Transform

An estimate of the original image f(x,y) can be obtained as

$$\hat{f}(r,\theta) = \int_0^{\pi}\int_{-\infty}^{\infty} |\omega| P_\phi(\omega) \exp[i\omega(x\cos\phi + y\sin\phi)]\, d\omega\, d\phi$$

$$= \int_0^{\pi} p_\phi^*(x')\, d\phi$$

where

$$p_\phi^*(x') = \int_{-\infty}^{\infty} |\omega| P_\phi(\omega) \exp(i\omega x')\, d\omega$$

$$= \mathscr{F}_1^{-1}\left[|\omega| P_\phi(\omega)\right]$$

$$= \mathscr{F}_1^{-1}[|\omega|] * p_\phi(x')$$

The Central Slice Theorem

and

$$P_\phi(\omega) = F(\omega\cos\phi, \omega\sin\phi)$$

$$= F(\omega_{x'}, \omega_{y'})\big|_\phi \quad \text{or} \quad F(\omega_x, \omega_y)\big|_\phi$$

$$= F(\omega, \phi)$$

Noise in Final Image – An Example

A few observations:

- The noise (r) on the final image is a direct consequence of the noise on the projection data.
- Every element in the final image is given as

$$\hat{f}_i = \sum_{j=1}^{m} s_{ij} \cdot g_j$$

- The reconstruction process is essentially a linear operator applied on the measured data $\boldsymbol{g}=[g_1, g_2, \ldots, g_M]$.

Where dose the noise on the reconstructed image come from?

Discrete Random Variable

- For a random variable that only takes on a discrete set of values, its distribution can be specified using the probability mass function (PMF)

$$\Pr[N = \eta_i], \text{ for } i = 1, 2, \ldots, k,$$

where $\Pr[N = \eta_i]$ is the probability that random variable N will take on the particular value η_i.

- The probability mass function (PMF) has the following properties

$$0 \le \Pr[N = \eta_i] \le 1, \quad \text{for } i = 1, 2, \ldots, k,$$

$$\sum_{i=1}^{k} \Pr[N = \eta_i] = 1,$$

$$P_N(\eta) = \Pr[N \le \eta] = \sum_{\text{all } \eta_i \le \eta} \Pr[N = \eta_i].$$

Independent Random Variable

- For a collection of multiple random variables $N_1, N_2, \ldots N_m,$

  Having the pdf's $p_1(\eta), p_2(\eta), \ldots, p_m(\eta),$

- The mean and variance of the sum of these variables are

$$\mu_S = \mu_1 + \mu_2 + \cdots + \mu_m\,,$$

$$\sigma_S^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2\,.$$

- And similarly

$$p_S(\eta) = p_1(\eta) * p_2(\eta) * \cdots * p_m(\eta)\,,$$

Covered in lecture

The Maximum Likelihood Reconstruction

Recall that the ***likelihood function,*** $L(\boldsymbol{f},\boldsymbol{g})$, of a possible source function $\boldsymbol{f}$ is

$$L(\mathbf{f},\boldsymbol{g}) = p(\boldsymbol{g} \mid \mathbf{f})$$

So that the maximum likelihood solution (the image that maximizing the likelihood function) can be found as

$$\hat{\mathbf{f}}_{ML} = \underset{\mathbf{f}}{\operatorname{argmax}}\, L(\mathbf{f},\boldsymbol{g})$$

or equivalently

$$\hat{\mathbf{f}}_{ML} = \underset{\mathbf{f}}{\arg\max} \log\left[L(\mathbf{f},\boldsymbol{g})\right] \equiv \underset{\mathbf{f}}{\operatorname{argmax}}\, l(\mathbf{f},\boldsymbol{g})$$

where $l(\mathbf{f},\boldsymbol{g})$ is the log-likelihood function

$$l(\mathbf{f},\boldsymbol{g}) = \log\left[L(\mathbf{f},\boldsymbol{g})\right]$$

Poisson Statistics of the Projection Data

The probability of a given projection data $\mathbf{g}=(g_1, g_2, g_3, \ldots, g_M)$ is

$$p(\mathbf{g}) = \prod_{m=1}^{M} p(g_m) = \prod_{m=1}^{M} \frac{\bar{g}_m^{\boxed{g_m}}}{g_m!} e^{-\bar{g}_m}$$

Measured no. of counts on detector pixel m.

where $\bar{g}_m$ is the expected value for the number of counts on detector pixel # m

$$\bar{g}_m = \sum_{n=1}^{N} f_n \boxed{p_{nm}}$$

In the context of emission tomography, p_{nm} is the probability of a gamma ray generated at a source pixel n is detected by detector element m.

Remember that

$$\begin{pmatrix} \bar{g}_1 \\ \bar{g}_2 \\ \bar{g}_3 \\ 7 \\ \bar{g}_M \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & 6 & p_{1N} \\ p_{21} & p_{22} & p_{23} & 6 & p_{2N} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3N} \\ 7 & 7 & 7 & 9 & 7 \\ p_{M1} & p_{M2} & p_{M3} & 6 & p_{MN} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{pmatrix}$$

The Maximum Likelihood Reconstruction

For data that follows Poisson distribution, the ***likelihood function,*** $L(\boldsymbol{f},\boldsymbol{g})$, of a possible source function $\boldsymbol{f}$ is

$$L(\mathbf{f},\mathbf{g}) = p(\mathbf{g} \mid \mathbf{f}) = \prod_{m=1}^{M} p(g_m) = \prod_{m=1}^{M} \frac{\overline{g}_m^{\,g_m}}{g_m!} e^{-\overline{g}_m}$$

$\overline{g}_m = \sum_{n=1}^{N} f_n p_{nm}$, is the expected number of counts on detector pixel m.

g_m is the measured number of counts on detector pixel m.

The log-likelihood function is

$$l(\mathbf{f},\mathbf{g}) = \log\left[L(\mathbf{f},\mathbf{g})\right] = \log\left[p(\mathbf{g} \mid \mathbf{f})\right]$$

$$= \prod_{m=1}^{M} p(g_m) = \log\left[\prod_{m=1}^{M} \frac{\overline{g}_m^{\,g_m}}{g_m!} e^{-\overline{g}_m}\right] = \sum_{m=1}^{M} \left(g_m \log \overline{g}_m - \overline{g}_m - \log g_m!\right)$$

The Maximum Likelihood Reconstruction

So the ML reconstruction for Poisson distributed data is

$$\hat{\boldsymbol{f}}_{ML} = \arg\max_{\boldsymbol{f}} l(\boldsymbol{f},\boldsymbol{g}) = \arg\max_{\boldsymbol{f}} \left[\sum_{m=1}^{M} \left(g_m \log \bar{g}_m - \bar{g}_m - \log g_m! \right) \right]$$

since g_m is not function of $\boldsymbol{f}$, we get

$$\boxed{\hat{\boldsymbol{f}}_{ML} = \arg\max_{\boldsymbol{f}} l(\boldsymbol{f},\boldsymbol{g}) = \arg\max_{\boldsymbol{f}} \left[\sum_{m=1}^{M} \left(g_m \log \bar{g}_m - \bar{g}_m \right) \right]}$$

The Maximum Likelihood Expectation Maximization (MLEM) Algorithm

The source (image) function ***f*** that has the maximum likelihood of giving rise to the observed data ***g*** can be found by the following iterative updating scheme

$$f_n^{(new)} = \frac{f_n^{(old)}}{\sum_{m=1}^{M} p_{nm}} \sum_{m=1}^{M} \frac{g_m}{\sum_{n'=1}^{N} p_{n'm} f_{n'}^{(old)}} p_{nm}, \quad n = 1,2,\ldots,N$$

where

$m = 1,2,\ldots,M$ is the index of detector elements in the imaging system

$n = 1,2,\ldots,N$ is the index of source object pixels

$f_n^{(old)}$: the current estimate of the source function in pixel n

$f_n^{(new)}$: the updated estimate of the source pixel n

p_{nm} : the probability of a gamma ray generated in source pixel n and detected by the detector element m

g_m : the observed number of counts in detector pixel m

Covariance of Random Variables

Covariance provides a measure of the strength of the correlation between two or more random variables. The covariance for two random variables and, each with sample size, is defined by the expectation value

$$Variance(N_1) \equiv \sigma^2 = \int_{N_1} p(N_1) \cdot (N, -\bar{N}_1)^2 \cdot dN_1$$

$$Covariance(N_1, N_2) = \iint_{N_1 N_2} p(N_1) \cdot p(N_2) \cdot (N, -\bar{N}_1) \cdot (N_2 - \bar{N}_2) \cdot dN_2 dN_2$$

Covered in lecture

Why Gaussian Random Variable is Important?

- When a quantity is derived as the result of a large number of accumulative effects, and each effect has a small contribution to the final outcome, then the distribution of the quantity tends to follow Gaussian distribution.

- The measured value on a detector is the result of accumulated interactions during an measurements session…
- The reconstructed image at a given pixel is the sum of the contributions from the data acquired with a large number of detector elements in the system…

$$\hat{f}_i = \sum_{j=1}^{m} s_{ij} \cdot g_j \qquad \textbf{or} \qquad \hat{\boldsymbol{f}} = \boldsymbol{S} \cdot \boldsymbol{g}$$

In this simple case, we know exactly what is the probability of a given reconstructed image!!

- It should follow Gaussian distribution

Mean of the reconstructed image: $\boldsymbol{f} = \hat{\boldsymbol{S}} \cdot \bar{\boldsymbol{g}}$

Covariance: $Cov\left(\hat{\boldsymbol{f}}\right) = \boldsymbol{S} \cdot Cov(\boldsymbol{g}) \cdot \boldsymbol{S}^T$

Covered in lecture

Signal-to-Noise Ratio (SNR)

- The Amplitude SNR

$$\mathrm{SNR_a} = \frac{\mathrm{Amplitude}(f)}{\mathrm{Amplitude}(N)} \,.$$

- There are many ways to define the amplitude SNR.
- The amplitude of the noise is normally referred to as the standard deviation of the random fluctuation associated with the signal.

- The Power SNR

$$\mathrm{SNR_p} = \frac{\mathrm{power}(f)}{\mathrm{power}(N)} \,.$$

- Again, the exact definition of the power SNR also depends on the definition of the signal power and noise power.

Magnitude and Phase

- In general, Fourier transform is a complex valued signal, even if f(x,y) is real valued.
- It is sometimes useful to consider the magnitude and phase of the Fourier transform separately.

Fourier coefficients are complex: $$F(u,v) = F_R(u,v) + j \cdot F_I(u,v)$$

Magnitude: $$|F(u,v)| = \sqrt{F_R^2(u,v) + F_I^2(u,v)}$$

Phase: $$\angle F(u,v) = \tan^{-1} \frac{F_I(u,v)}{F_R(u,v)}$$

An alternative representation: $$F(u,v) = |F(u,v)| e^{j\angle F(u,v)}$$

- The square of the magnitude $|F(u,v)|^2$ is referred to as the power spectrum of the original function.

Covered in lecture

Signal-to-Noise Ratio (SNR)

- Example 1: Suppose that (a) the noise at any given location is independent to the noise at other locations and (b) the mean and variance of the noise is constant,

$$\mathrm{SNR_p} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x,y) * f(x,y)|^2 \, dx \, dy}{\sigma_N^2},$$

$$\mu_N(x,y) = 0 \qquad \text{and} \qquad \sigma_N(x,y) = \sigma_N.$$

Remember that with linear reconstruction, the final image is

$$\hat{f}_i = \sum_{j=1}^{M} q_{ij} g_j$$

So reconstructed image at different points came from the same set of data (g_1, g_2, …, g_M).

they are not independent!

- This represents the most simple form of noise, constant variance and no correlation between noise values at different locations.

Covered in lecture

Signal-to-Noise Ratio (SNR)

- Example 2: Let' s consider a wide-sense stationary (WSS) noise.
- For a more realistic noise model, we should consider that the noise at different location in the image is correlated.

$$\text{Constant mean}: E\left[N(x,y)\right] = \mu_N$$

$$\text{Covariance:} E\left\{\left[N(x_1,y_1)-\mu_N\right]\cdot\left[N(x_2,y_2)-\mu_N\right]\right\} = R\left(\tau_1,\tau_2\right)$$

$$\textit{where} \quad \tau_1 = x_1 - x_2, \tau_2 = y_1 - y_2$$

$$\textit{and } N(x,y) \text{ is the noise value at a given location } (x,y).$$

Covered in lecture

Signal-to-Noise Ratio (SNR)

- For this case, the noise power spectrum is

$$\mathrm{NPS}(u,v) = \lim_{x_0,y_0\to\infty} \frac{1}{4x_0y_0}\mathrm{E}\left[\left|\int_{-x_0}^{x_0}\int_{-y_0}^{y_0}[N(x,y)-\mu_N]\right.\right.$$
$$\left.\left.\exp\left(-j2\pi(ux+vy)\right)\,dxdy\right|^2\right],$$

- And the power SNR is

$$\mathrm{SNR_p} = \frac{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|h(x,y)*f(x,y)|^2\,dxdy}{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{NPS}(u,v)\,dudv},$$

Covered in lecture

Properties of Fourier Transform (4)

- Parseval's Theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x,y)\right|^2 dudv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F(u,v)\right|^2 dudv$$

Total energy of a signal of a signal f(x,y) in spatial domain equals its total energy in spatial frequency domain.

Fourier transform and its inverse is energy preserving!

Covered in
Covered in lecture

Signal-to-Noise Ratio (SNR)

- By using the Parseval's theorem, we have

$$\mathrm{SNR_p} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |H(u,v)|^2\ |F(u,v)|^2\, du dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{NPS}(u,v)\, du dv},$$

This provides the relationship between contrast, resolution, noise etc.

$$= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{SNR_p}(u,v)\mathrm{NPS}(u,v)\, du dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{NPS}(u,v)\, du dv},$$

- And then with the definition of MTF, we have the frequency-dependent power SNR:

$$\mathrm{SNR_p}(u,v) = \frac{|H(u,v)|^2 |F(u,v)|^2}{\mathrm{NPS}(u,v)} = \frac{\mathrm{MTF}^2(u,v)}{\mathrm{NPS}(u,v)}\ |F(u,v)|^2$$

Review of the Key Concepts

Contrast

- What is contrast?
- How to quantify contrast?
- What is the modulation transform function (MFT)?

Spatial Resolution

- Definition of resolution
- Commonly used measures of resolution
- Relations between MTF and resolution

Image Noise

- Random variables
- Origin of the noise in image
- How to quantify image noise in a linear system?

Signal-to-noise ratio (SNR)

- Amplitude and power SNR.