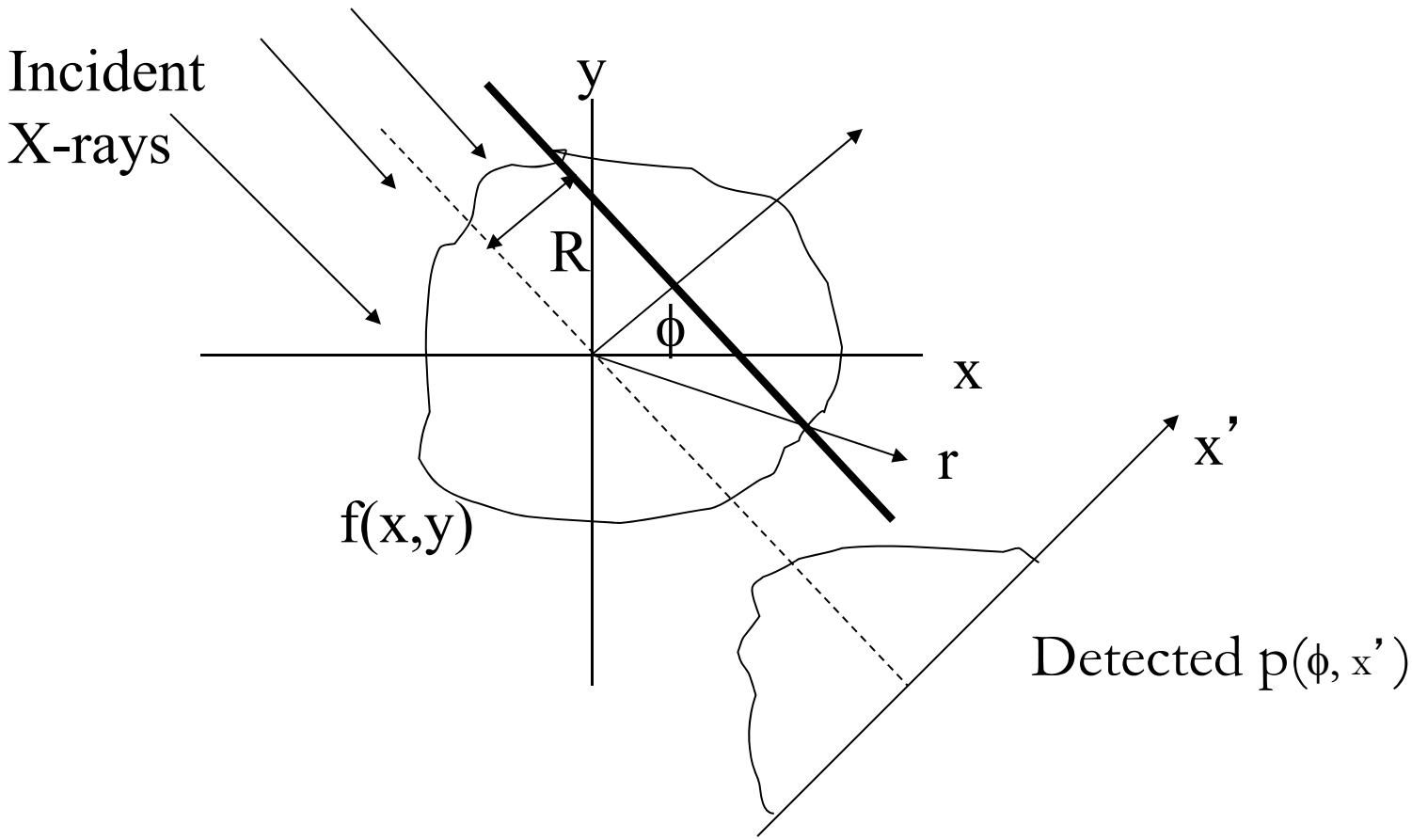
Central Section Or Projection Slice Theorem

$$F\{p(\phi, x')\} = F(r,\phi)$$

So in words, the Fourier transform of a projection at angle ϕ gives us a line in the polar Fourier space at the same angle ϕ.

Central slice theorem is the key to understand reconstructions from projection data

Central Slice Theorem



The thick line is described by

$$x \cos \phi + y \sin \phi = R$$

Central Slice Theorem

The projection $g_{\phi}(x')$ can thus be calculated as a set of line integrals, each at a unique x'.

$$p(\phi, x') = \int_{\infty}^{\infty} \int f(x,y)\, \delta(x \cos \phi + y \sin \phi - x')\, dx\, dy$$

Or alternatively,

$$p(\phi, x') = \int_{0}^{\infty} \int_{0}^{2\pi} f(r, \theta)\, \delta(r \cos(\phi - \theta) - x')\, r\, dr\, d\theta$$

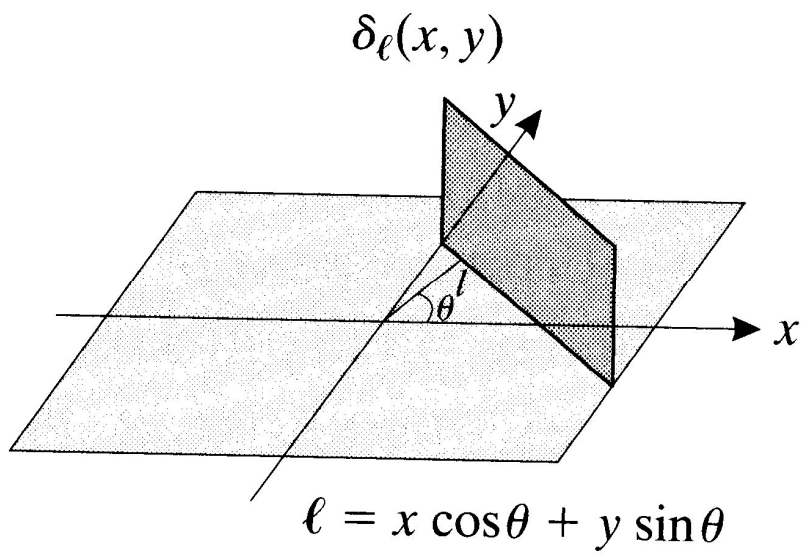
in polar coordinates.

Again $p(\phi, x')$ can be treated as a 1D function of x' at a given angle ϕ.

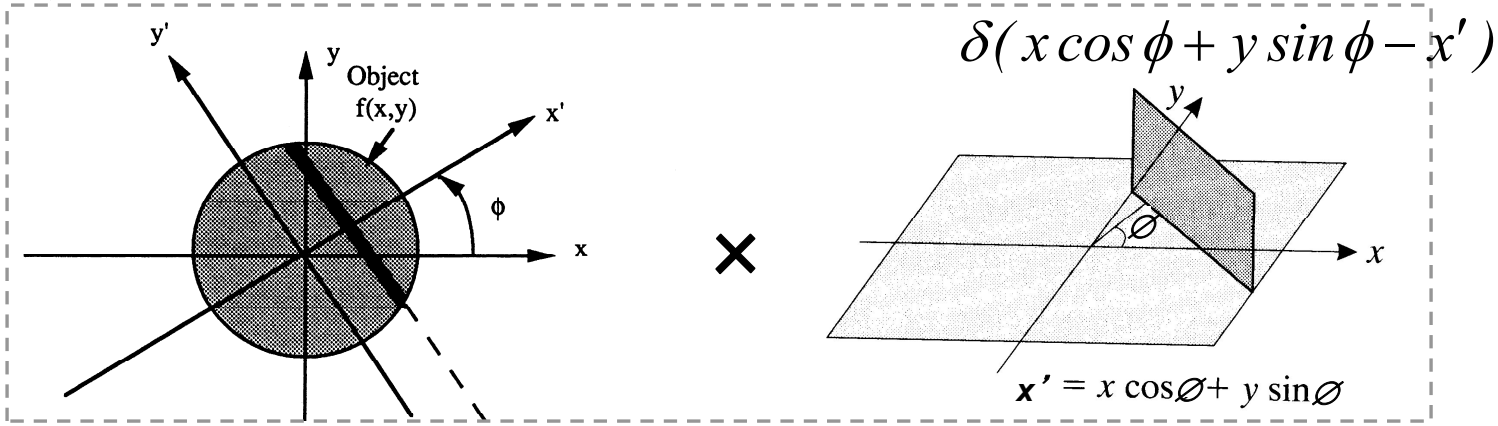
Line Impulse Signal (1)

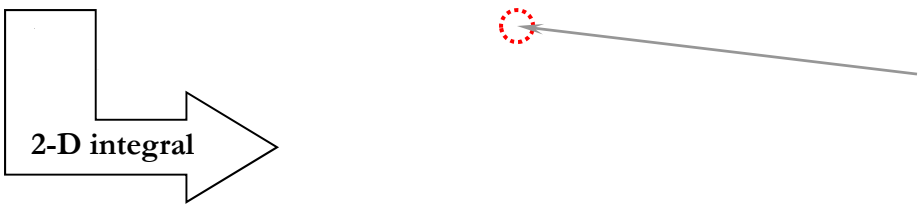
$$\delta_L(x,y) = \delta(x\cos\theta + y\sin\theta - l)$$

$$where\ \delta(x) = \begin{cases} > 0, & x\cos\theta + y\sin\theta = l \\ 0, & otherwise \end{cases}$$



Line Impulse Signal and Radon Transform



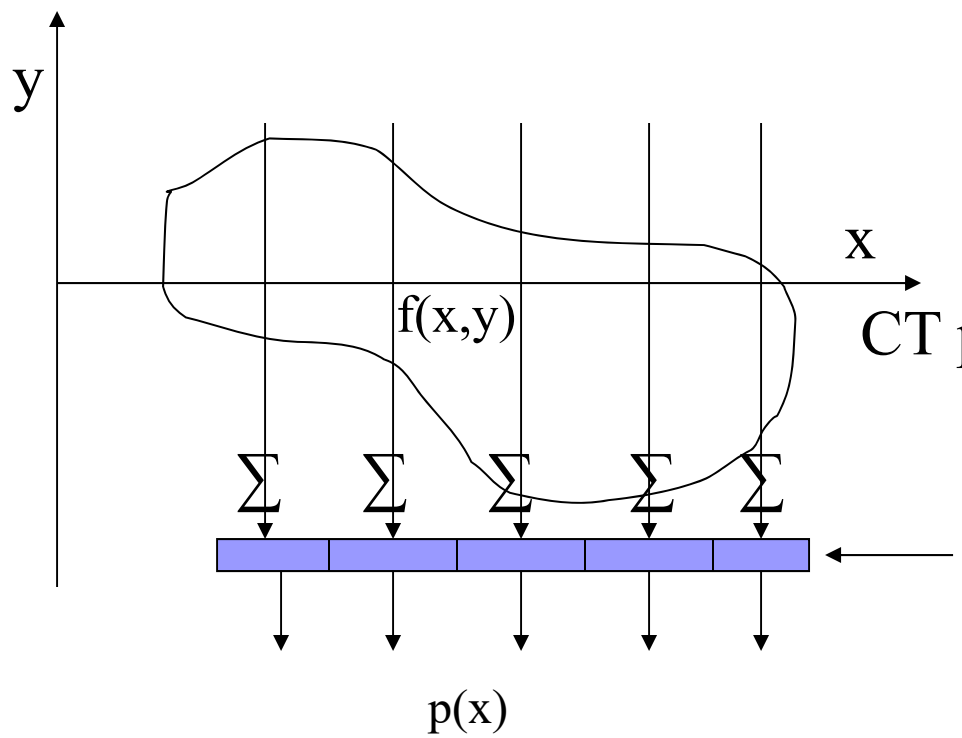


The value of the projection function $p_\phi(x')$ at this point is the integral of the function of f(x,y) along the straight line: $x' = x\cos\phi + y\sin\phi$

The integral of a line impulse function and a given 2-D signal gives the ***projection*** data from a given view …

$$p(\phi, x') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\phi + y\sin\phi - x')dxdy$$

X-ray Projection Revisited



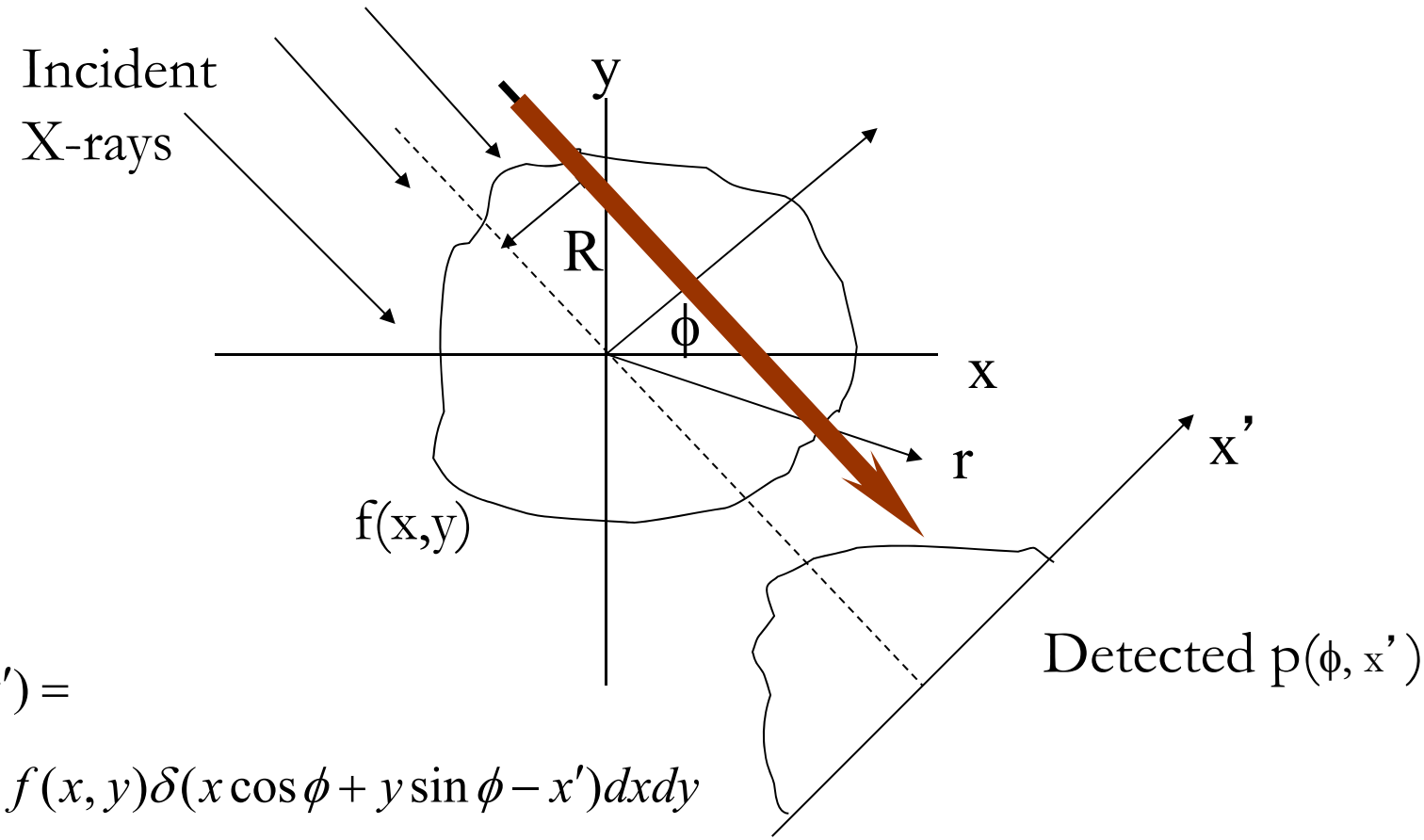
CT projection measure line integrals

p(x) = CT detector array output

$$p(x) = \int_{y=-\infty}^{y=\infty} f(x,y)\,dy$$

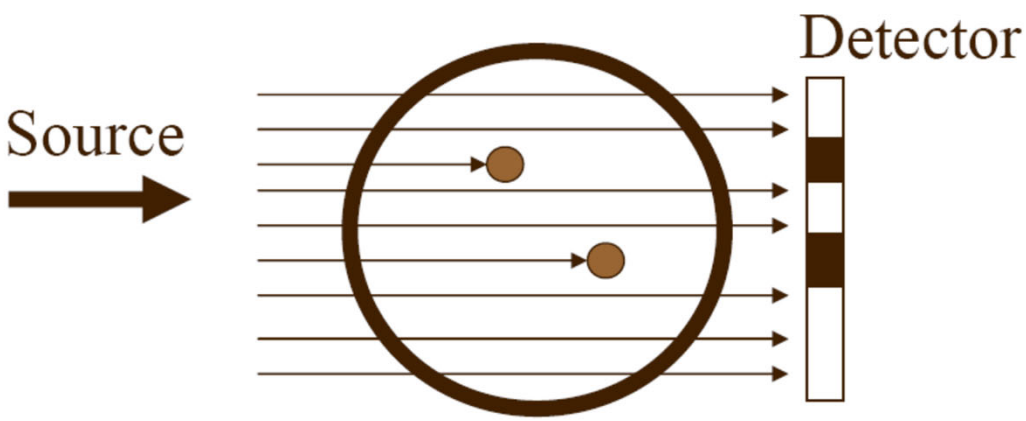
Review of 2-D Analytical Reconstruction Methods
Projection Data

Projection data p(ϕ, x')



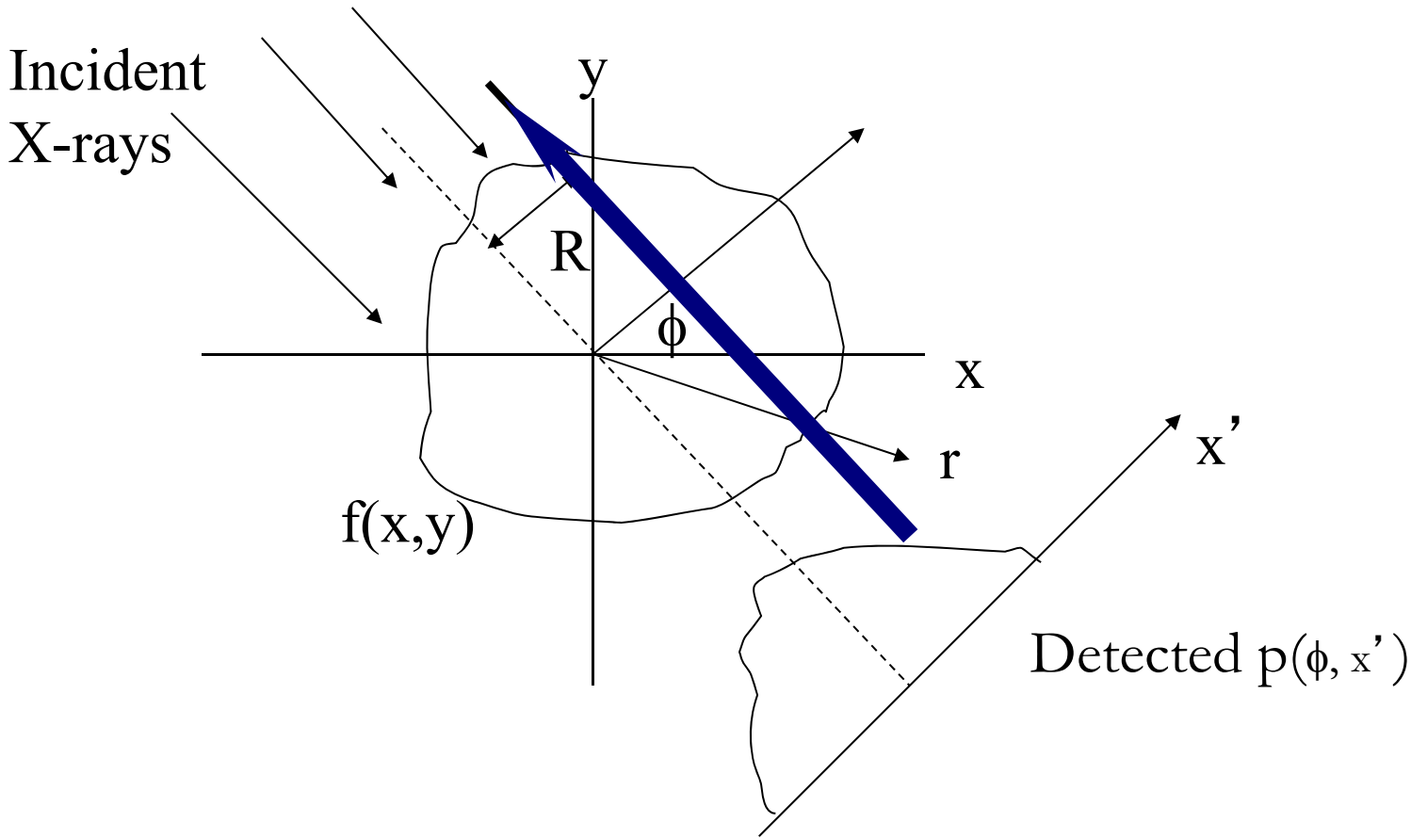
$$p(\phi, x') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\phi + y\sin\phi - x')dxdy$$

Simple Backprojection



inverse = 1 back projection

Review of 2-D Analytical Reconstruction Methods
Back Projection Operation



Simple Backprojection

Crude Idea 1: Take each projection and smear it back along the lines of integration it was calculated over.

Result from a back projection from a single view angle:

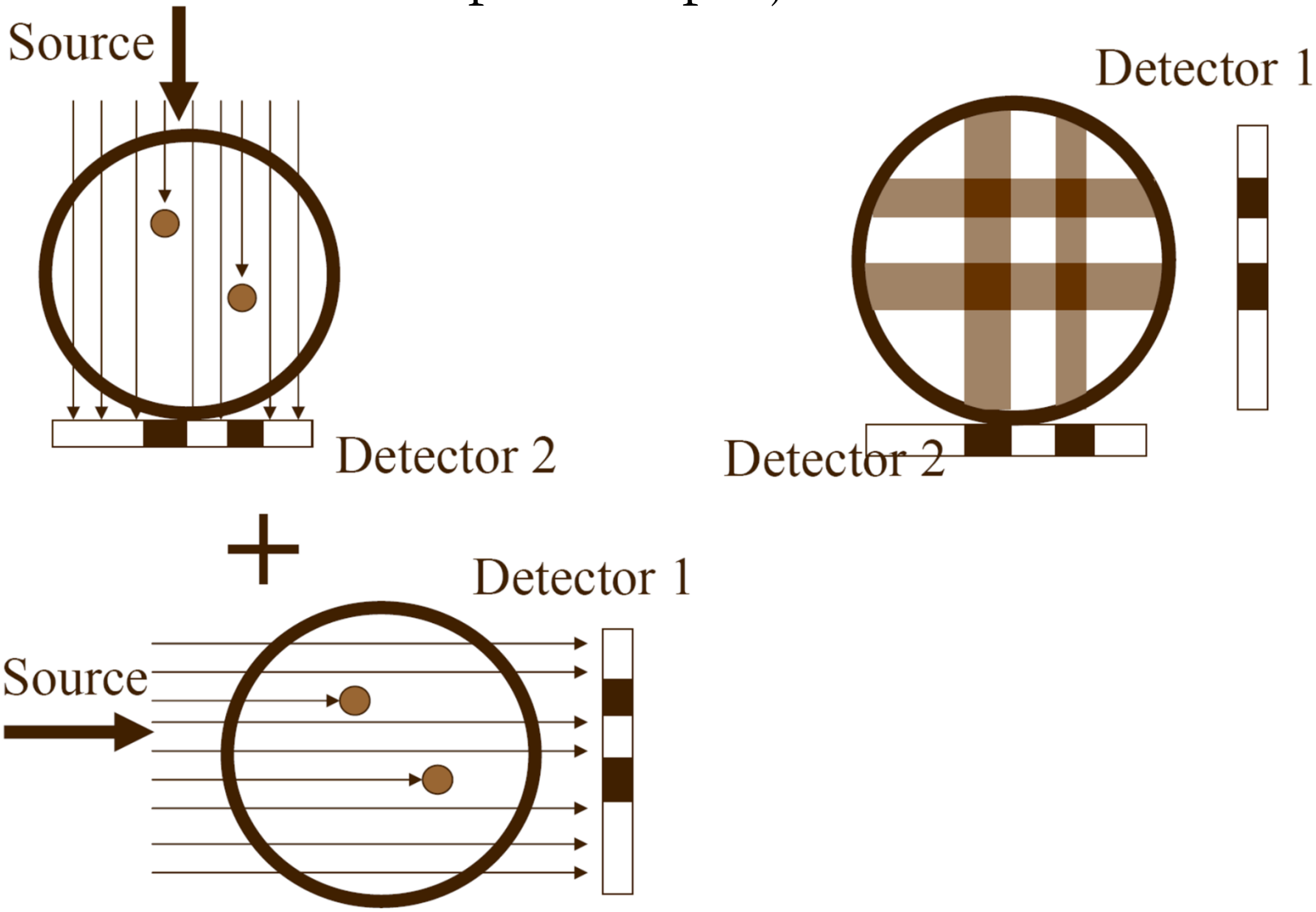
$$b_\phi(x,y) = \int p_\phi(x') \, \delta(x \cos\phi + y \sin\phi - x') \, dx'$$

Adding up all the back projections from all the angles gives,

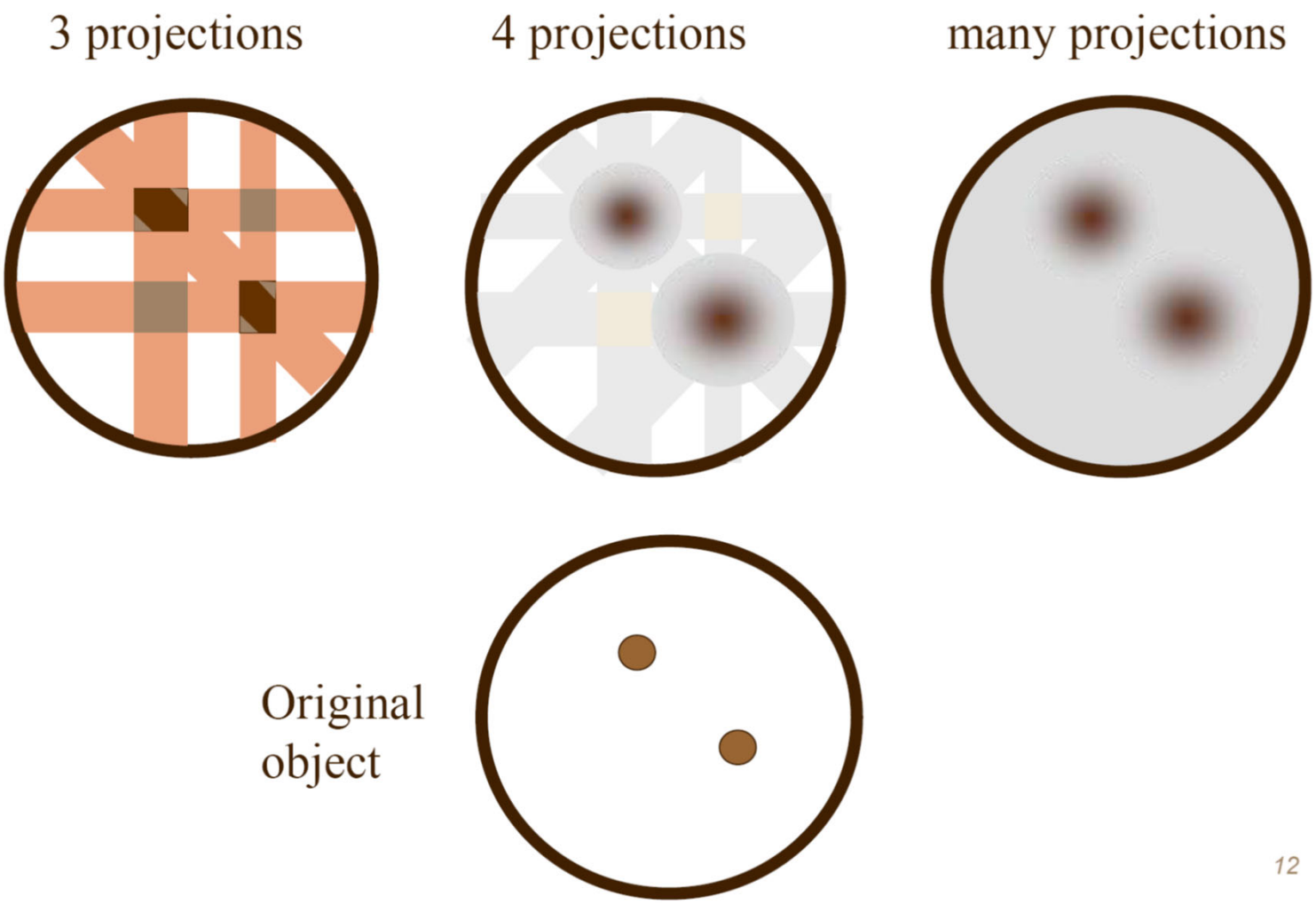
$$f_{back\text{-}projection}(x,y) = \int b_\phi(x,y) \, d\phi$$

$$f_{\text{back-projection}}(x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} p_\phi(x') \, \delta(x \cos\phi + y \sin\phi - x') \, dx'$$

Simple Backprojection



Simple Backprojection



Simple Back-projection and the 1/r Blurring

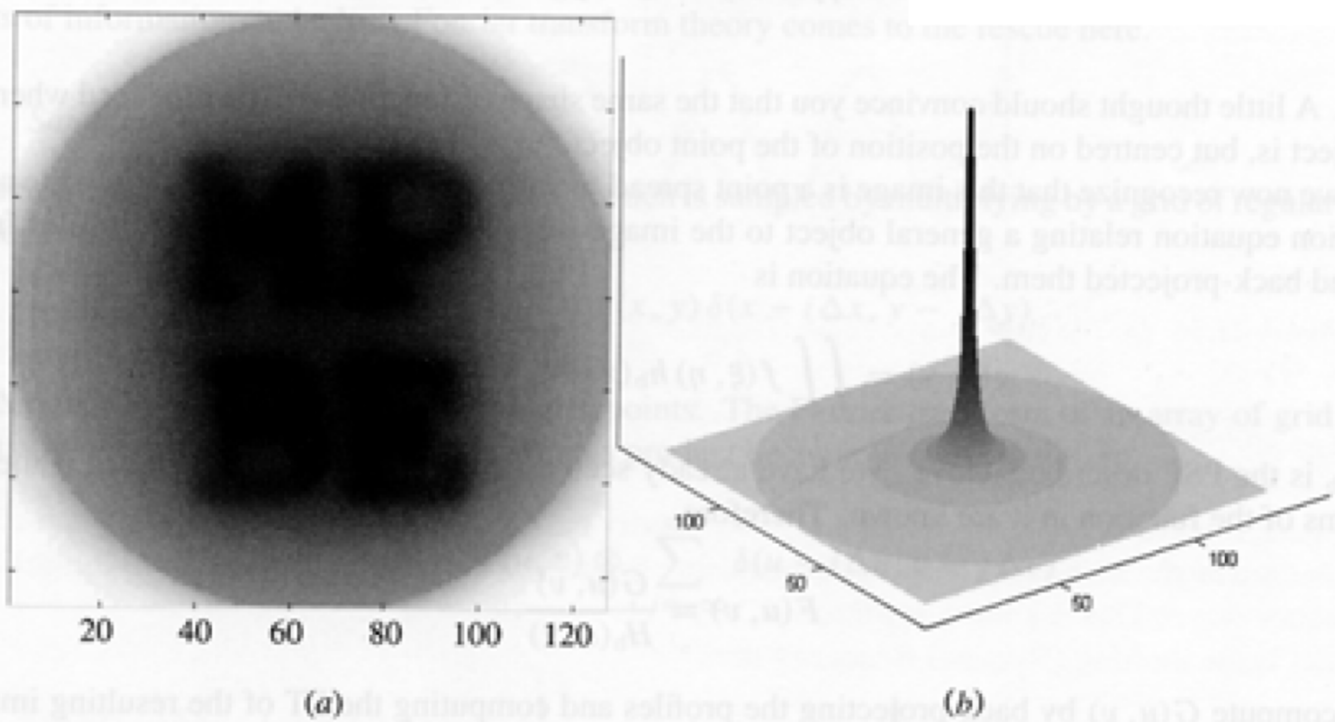


Figure 11.14. *(a) Image produced by back-projecting the sinogram given in figure 11.12(b); (b) the response to a point object in the middle of the field of view.*

From Medical Physics and Biomedical Engineering, Brown, IoP Publishing

Impulse Response Function of Simple Backprojection Operator

$h_b(r) = 1/r$

$f_b(x,y) = f(x,y) * 1/r$

$F_b(\rho,\phi) = F(\rho, \phi) / \rho$, since $F\{1/r\} = 1/\rho$

Back projected image is blurred by convolution with 1/r

Simple Back-projection and the 1/r Blurring

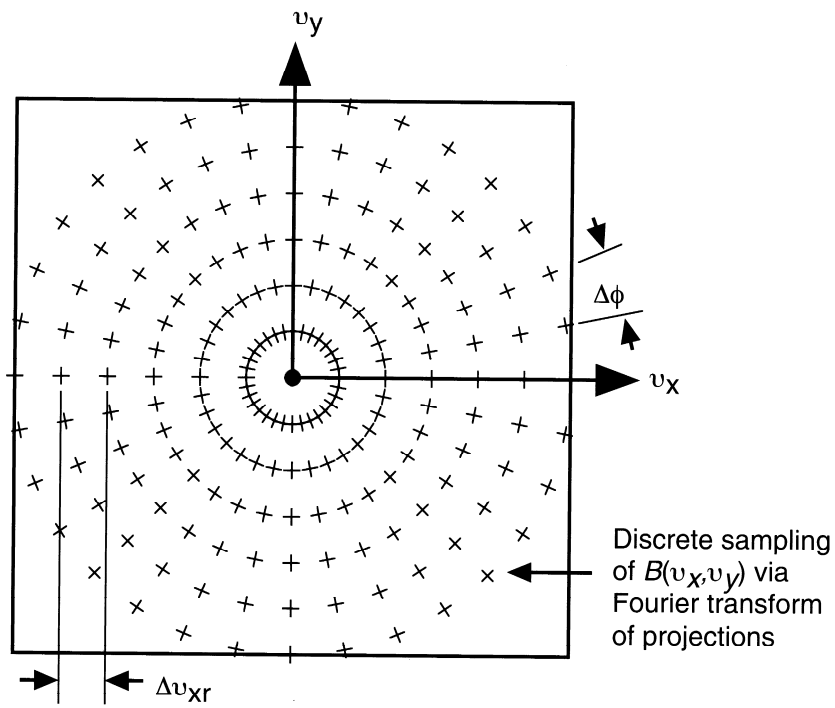


FIGURE 18 The discrete sampling pattern of $F(\upsilon_x,\upsilon_y)$ contained in $B(\upsilon_x,\upsilon_y)$, resulting from the use of discretely sampled projections.

The nature of the 1/r blurring:
Radon transform produced equally spaced radial sampling in Fourier domain.

Simple Backprojection

Crude Idea 1: Take each projection and smear it back along the lines of integration it was calculated over.

Result from a back projection from a single view angle:

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Adding up all the back projections from all the angles gives,

$$f_{back\text{-}projection}(x,y) = \int b_\phi(x,y) \, d\phi$$

$$f_{\text{back-projection}}(x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} p_\phi(x') \, \delta(x \cos\phi + y \sin\phi - x') \, dx'$$

The Nature of the 1/r Blurring

The nature of the 1/r blurring:

Radon transform produced equally spaced radial sampling in Fourier domain.
The low frequency components are over sampled, which causes

$h_b(r) = 1/r$ *and* $f_b(x,y) = f(x,y) * 1/r$ in spatial domain

and

$F_b(\rho,\phi) = F(\rho, \phi) / \rho$ in spatial frequency domain

Inverse Radon Transform

Suppose the sample projection data preserves all information contained in the original function f(x,y), can we recover the exact function f(x,y) with an ***Inverse Radon Transform***?

$$\mathscr{R}^{-1}\{\mathscr{R}[f(x,y)]\} = \hat{f}(x,y)$$

Inverse Radon Transform

The estimate of the original image f(x,y) can be obtained as

$$\hat{f}(r,\theta) = \int_0^{\pi}\int_{-\infty}^{\infty} |\omega| P_\phi(\omega) \exp[i\omega(x\cos\phi + y\sin\phi)]\, d\omega\, d\phi$$

$$= \int_0^{\pi} p_\phi^*(x')\, d\phi$$

where

$$p_\phi^*(x') = \int_{-\infty}^{\infty} |\omega| P_\phi(\omega) \exp(i\omega x')\, d\omega$$

$$= \mathscr{F}_1^{-1}\left[|\omega| P_\phi(\omega)\right]$$

$$= \mathscr{F}_1^{-1}[|\omega|] * p_\phi(x')$$

and

The Central Slice Theorem

$$P_\phi(\omega) = F(\omega\cos\phi, \omega\sin\phi)$$

$$= F(\omega_{x'}, \omega_{y'})\big|_\phi \quad \text{or} \quad F(\omega_x, \omega_y)\big|_\phi$$

$$= F(\omega, \phi)$$

Filtered Back-projection

Mathematically, we can define a ***filtered back-projection*** FBP operation to remove the 1/r blurring.

$$\hat{f}(x, y) = \frac{1}{\pi}\int_0^{\pi} d\phi \int_{-\infty}^{\infty} dx'\, p_\phi(x')h(x\cos\phi + y\sin\phi - x')$$

where

$$h(x) = \mathscr{F}_1^{-1}[|\omega|]$$
$$= \mathscr{F}_1^{-1}[H(\omega)]$$

Can this be realized ??

Due to the diverging nature of the |w| function, the corresponding filter kernel does not exist in spatial domain!

Inverse Radon Transform

The estimate of the original image f(x,y) can be obtained as

$$\hat{f}(x,y) = \hat{f}(r,\theta) = \beta\mathscr{H}\{p_\phi(x')\}$$

The simple back projection operator

$$= \beta\mathscr{H}\{\mathscr{R}[f(x,y)]\}$$

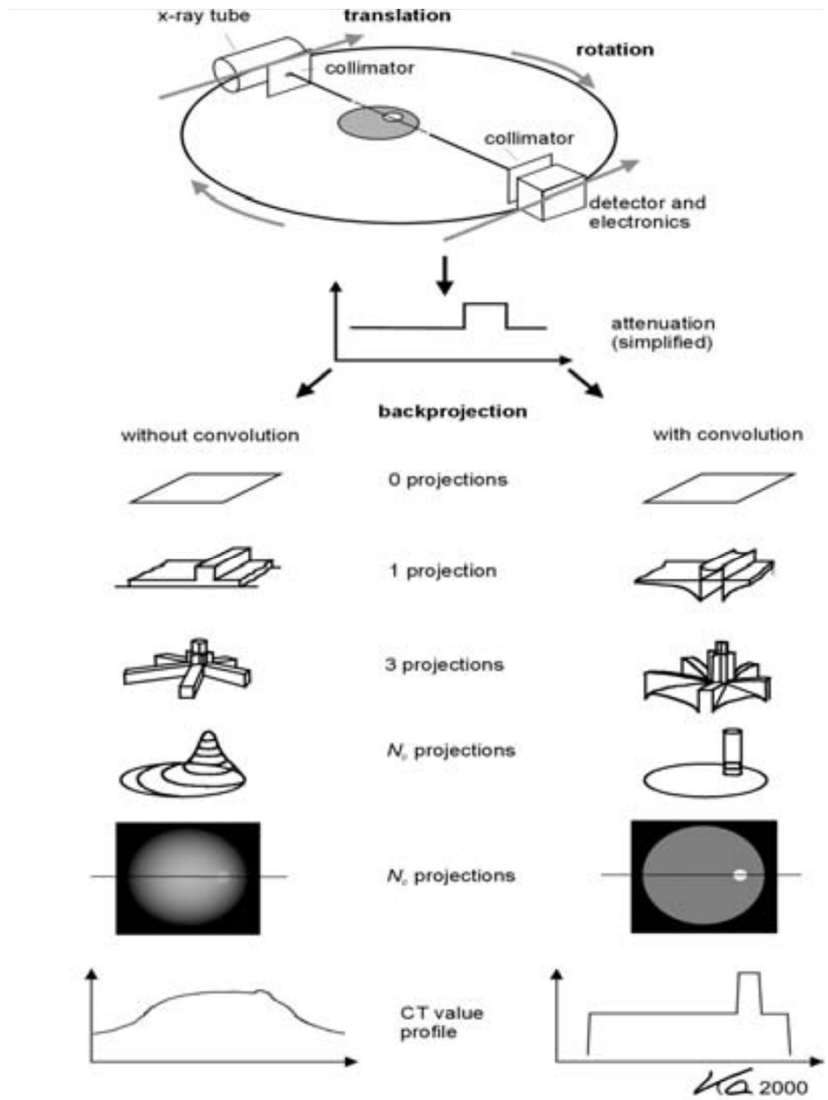
$$= \mathscr{R}^{-1}\{\mathscr{R}[f(x,y)]\}$$

where

$$\mathscr{H}\{p_\phi(x')\} = \mathscr{F}_1^{-1}[|\omega|] * p_\phi(x')$$

The inverse Radon transform can be represented as a filtering process followed by a back-projection operation.

Simple and Filtered Back-projection



From Computed Tomography, Kalender, 2000.

Simple and Filtered Back-projection

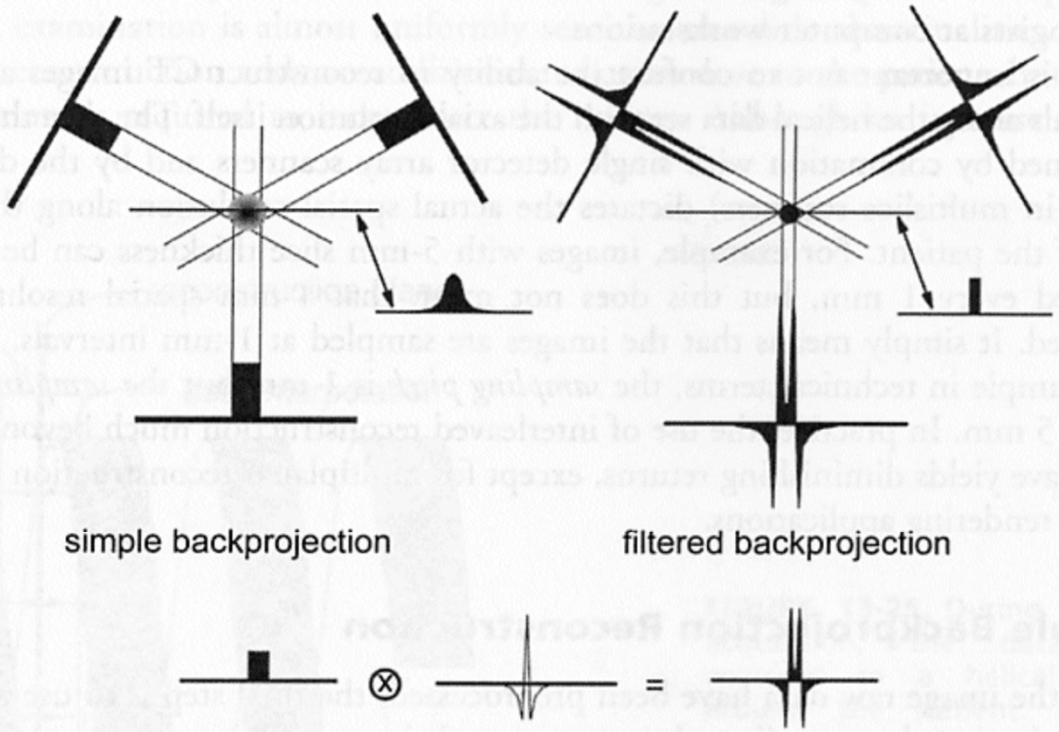
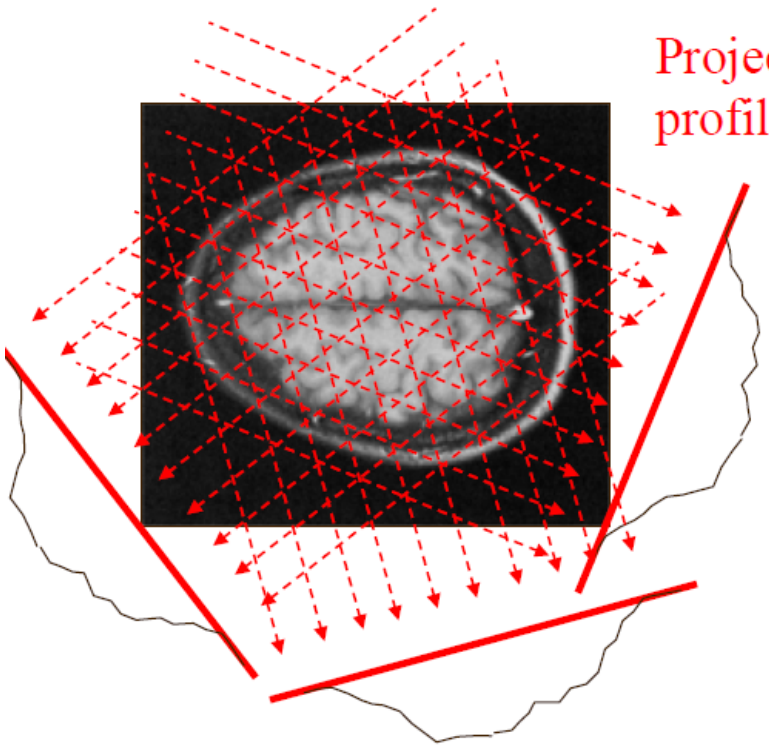


FIGURE 13-28. Simple backprojection is shown on the left; only three views are illustrated, but many views are actually used in computed tomography. A profile through the circular object, derived from simple backprojection, shows a characteristic 1/r blurring. With filtered backprojection, the raw projection data are convolved with a convolution kernel and the resulting projection data are used in the backprojection process. When this approach is used, the profile through the circular object demonstrates the crisp edges of the cylinder, which accurately reflects the object being scanned.

Chapters 12 & 13, The Essential Physics of Medical Imaging, Bushberg

Filtered Backprojection (FBP), What and why?

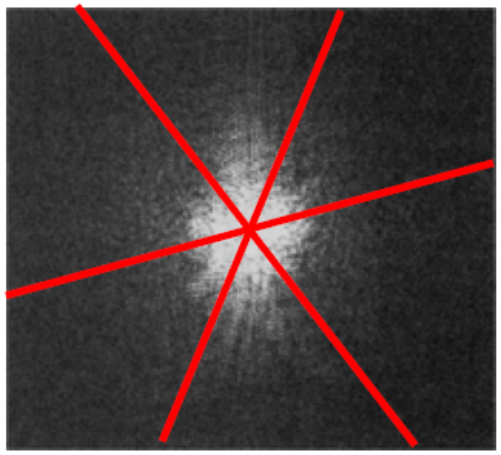
Central Slice Theorem



Projection profiles

1-D DFTs

Create lines in central slice of entire DFT image



Integrate intensities along x-direction

The more angles used, the better the Fourier space image is filled

http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf

One would need to use filter to compensate for this effect.

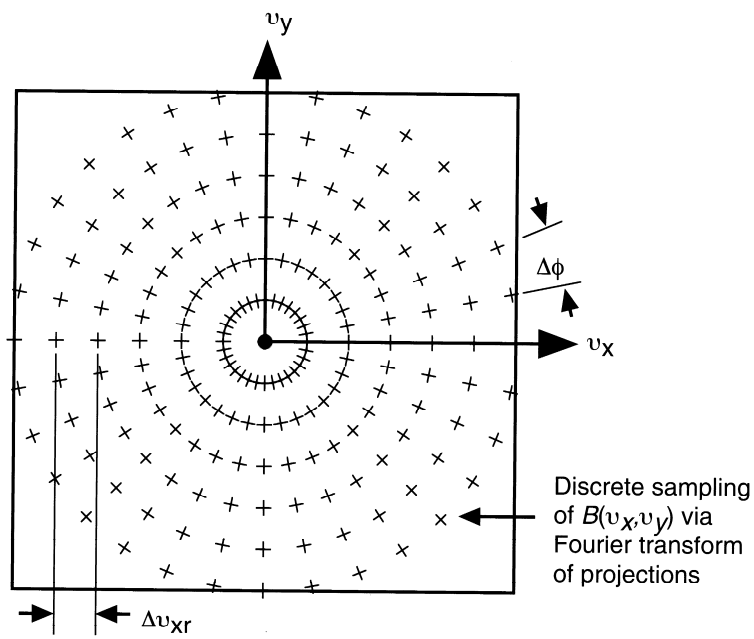


FIGURE 18 The discrete sampling pattern of $F(\upsilon_x, \upsilon_y)$ contained in $B(\upsilon_x, \upsilon_y)$, resulting from the use of discretely sampled projections.

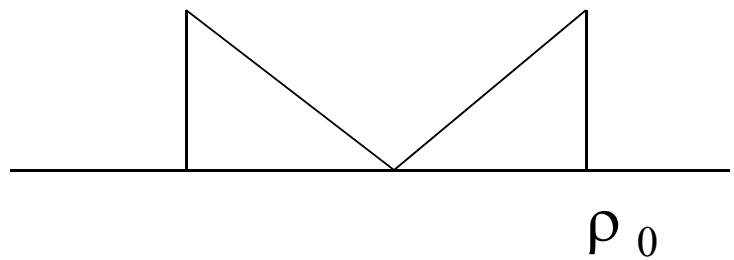
Ideally, one would use a perfect filter as in the Inverse Radon Transform

Filtered Back-projection

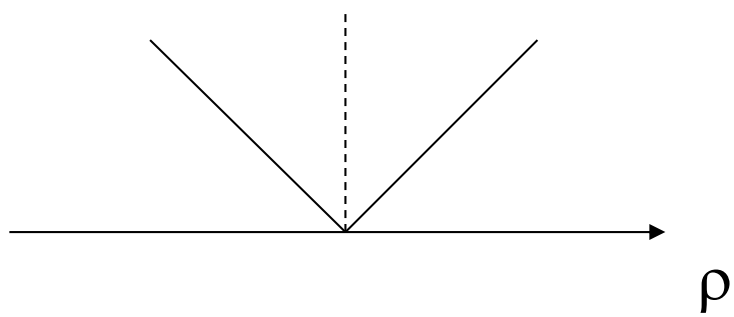
The Ram-Lak filter

$$H_{\mathrm{RL}}(\omega) = \begin{cases} |\omega|, & (|\omega| \le 2\pi B) \\ 0, & (\text{otherwise}) \end{cases}$$

Ram-Lak filter



Ideal filter



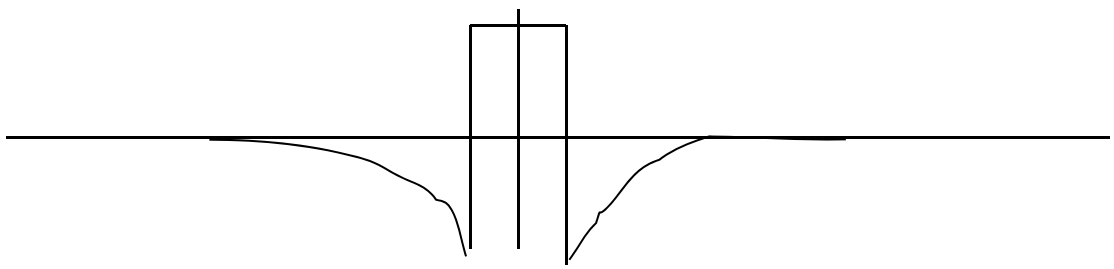
Consider the bandwidth-limited nature of most projection data, we have …

Filtered Back-projection

$$H_{\mathrm{RL}}(\omega) = \begin{cases} |\omega|, & (|\omega| \le 2\pi B) \\ 0, & (\text{otherwise}) \end{cases}$$

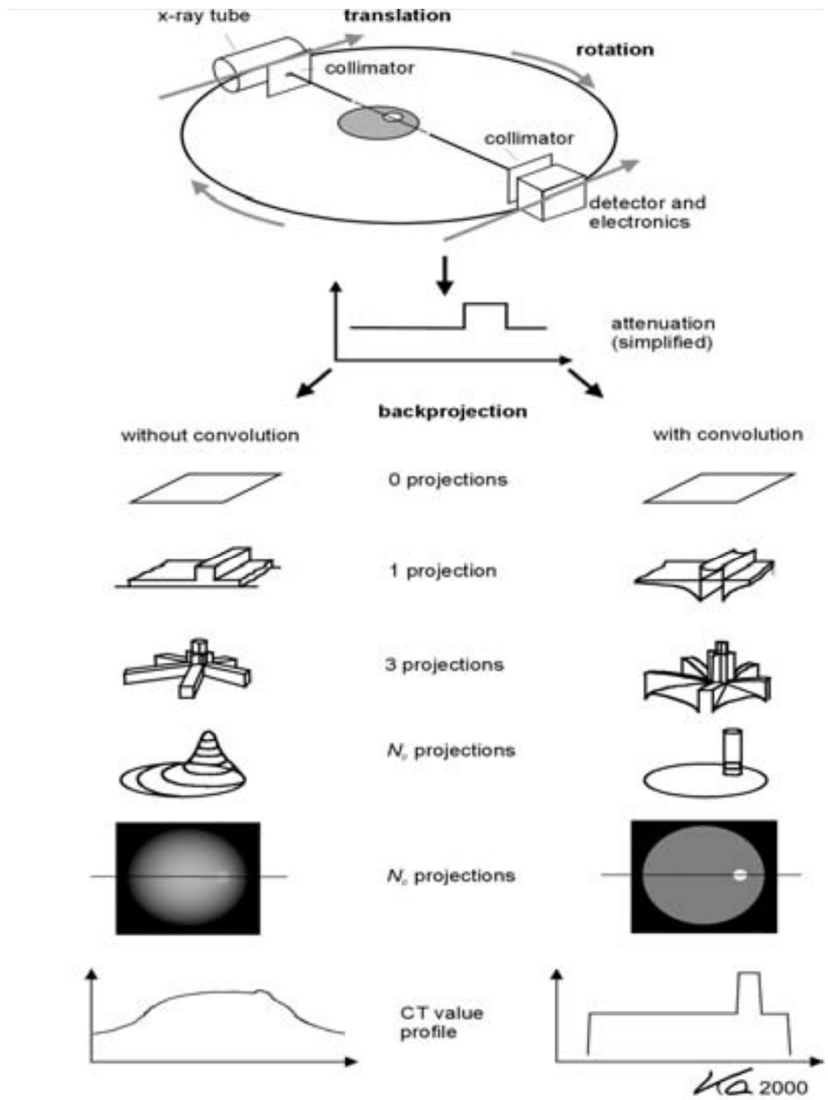
The Ram-Lak filter in spatial domain

$$\begin{aligned} h_{\mathrm{RL}}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{\mathrm{RL}}(\omega)\exp(ix\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} |\omega|\exp(ix\omega)\,d\omega \\ &= 2B^2\,\mathrm{sinc}(2\pi Bx) - B^2\,\mathrm{sinc}^2(\pi Bx) \end{aligned}$$



$$p_\phi^*(x') = \int_{-\infty}^{\infty} dx'\, p_\phi(x')h(x\cos\phi + y\sin\phi - x')$$

Simple and Filtered Back-projection



From Computed Tomography, Kalender, 2000.

Simple and Filtered Back-projection

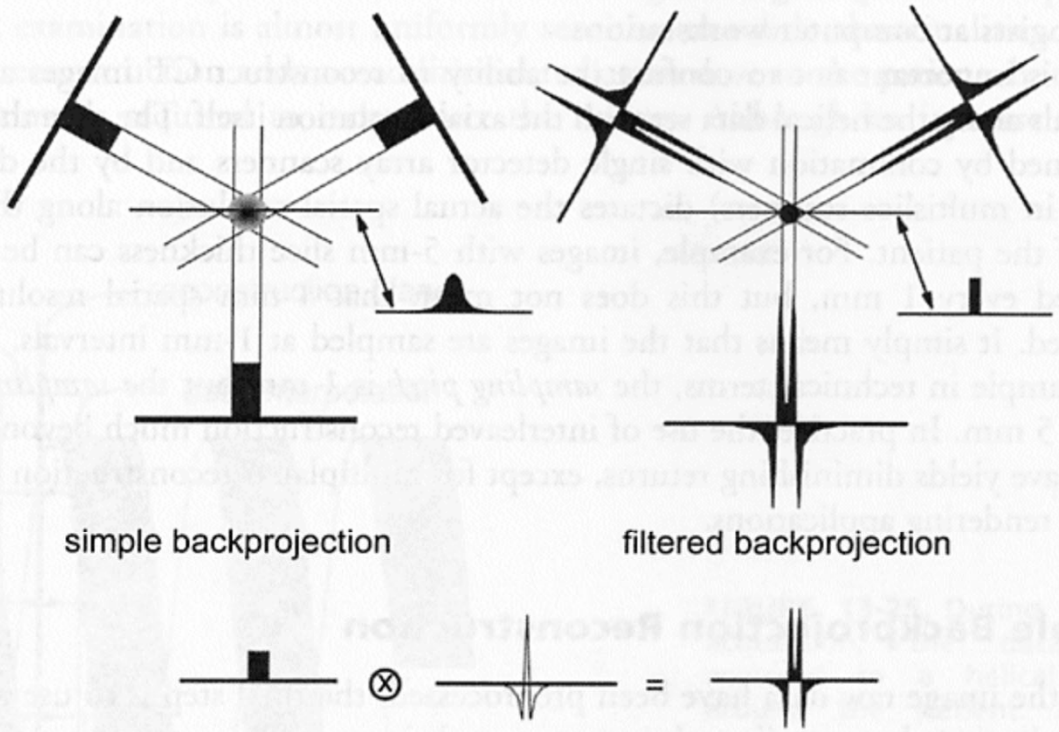
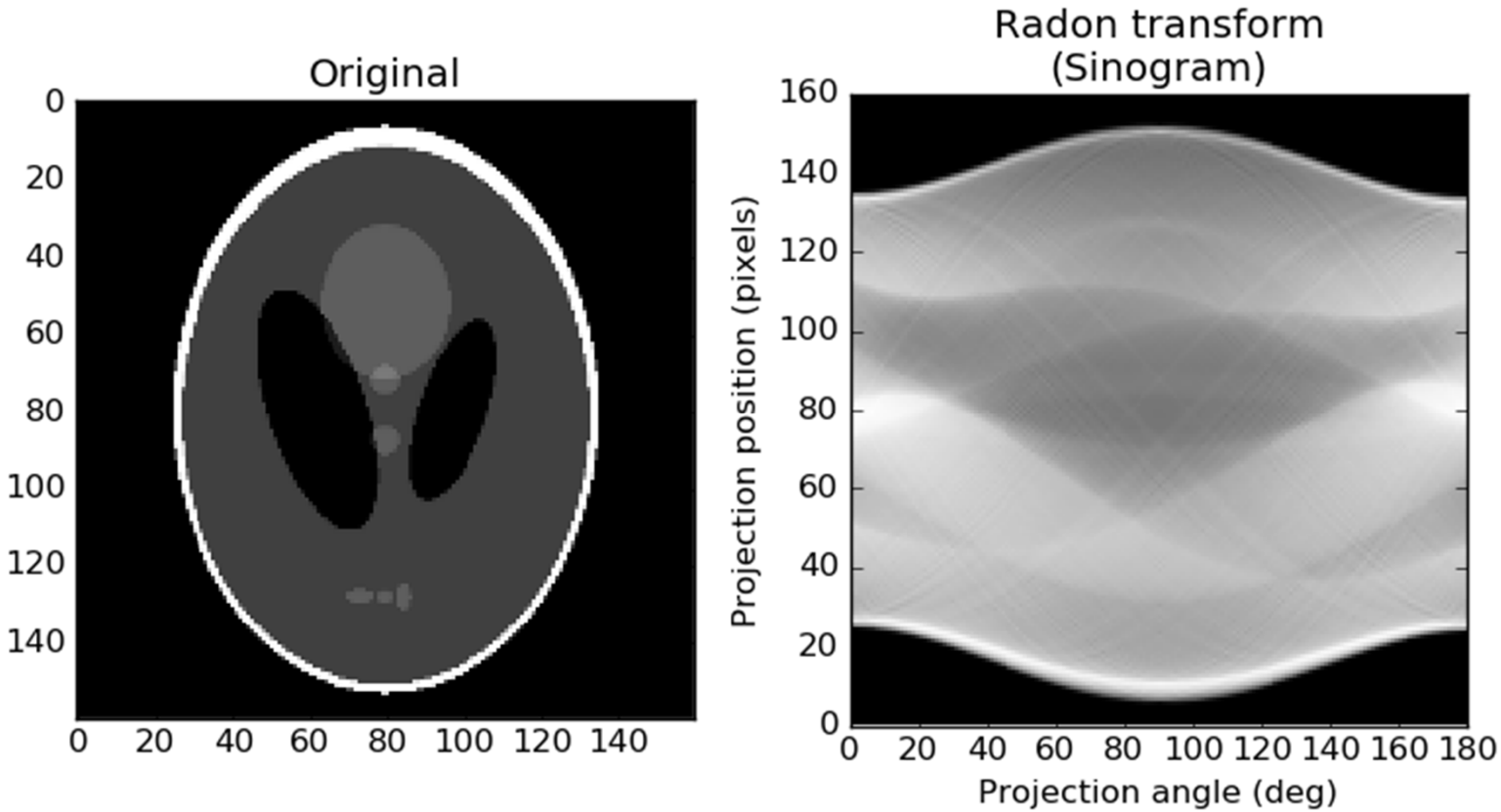


FIGURE 13-28. Simple backprojection is shown on the left; only three views are illustrated, but many views are actually used in computed tomography. A profile through the circular object, derived from simple backprojection, shows a characteristic 1/r blurring. With filtered backprojection, the raw projection data are convolved with a convolution kernel and the resulting projection data are used in the backprojection process. When this approach is used, the profile through the circular object demonstrates the crisp edges of the cylinder, which accurately reflects the object being scanned.

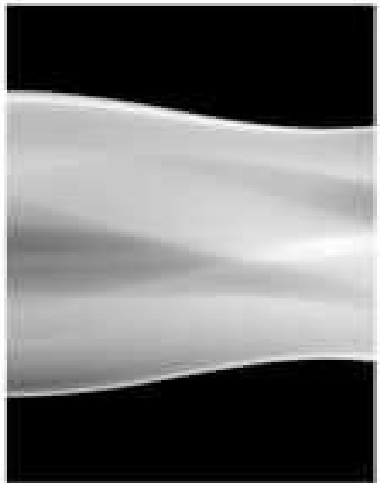
Chapters 12 & 13, The Essential Physics of Medical Imaging, Bushberg

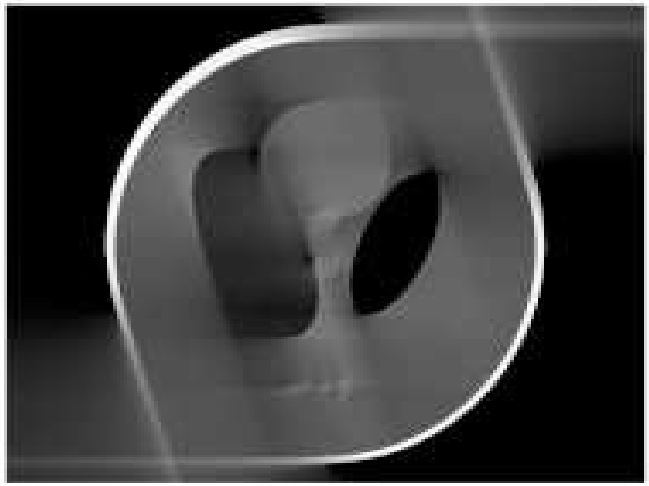
Radon Transform and Sinogram



http://tech.snmjournals.org/cgi/content-nw/full/29/1/4/F3

Filtered Back-projection





https://www.youtube.com/watch?v=ddZeLNh9aac

But is there something missing in this discussion?, such as

Noise in data?

Possible artifacts?

Filtered Back-projection – The Optimum Filter

Have we forgot something? What about the noise in the projection data?

Where is the noise coming from?

In reality, the true projection is

$$\dot{p}_\phi(x') = p_\phi(x') + n_\phi(x')$$

Review of Fourier Transform and Filtering Spectral Filtering

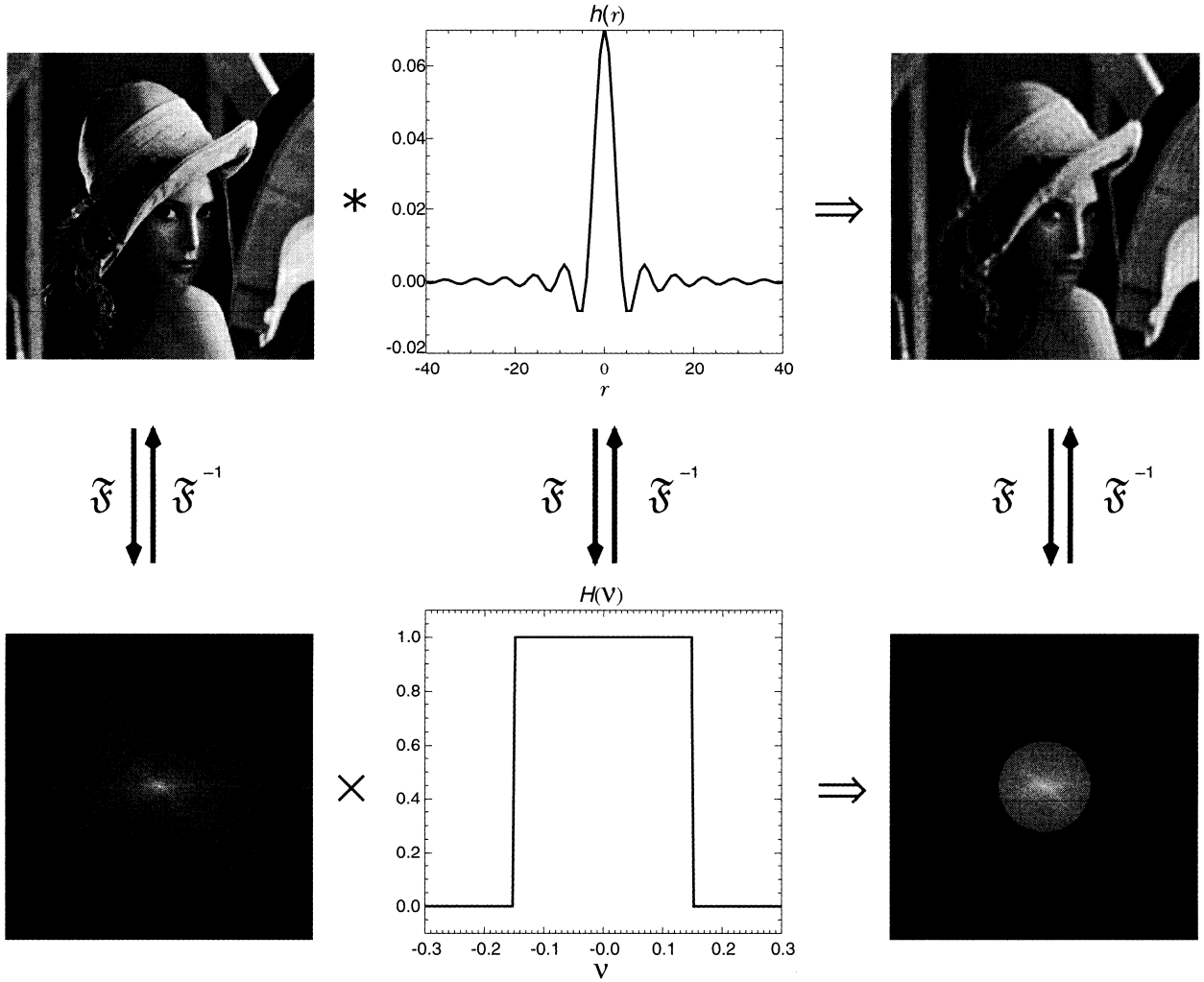


FIGURE 3 Ideal lowpass filtering of an image. Figures in the top and bottom rows demonstrate the filtering operation in the spatial and frequency domains, respectively. The left column shows the input image and the magnitude of its Fourier transform. The middle column are the circular symmetric PSF and its transfer function, where $\nu = \sqrt{\nu_x^2 + \nu_y^2}$ and $r = \sqrt{x^2 + y^2}$. Images in the right column are the output image and the magnitude of its Fourier transform.

Review of Fourier Transform and Filtering
Spectral Filtering

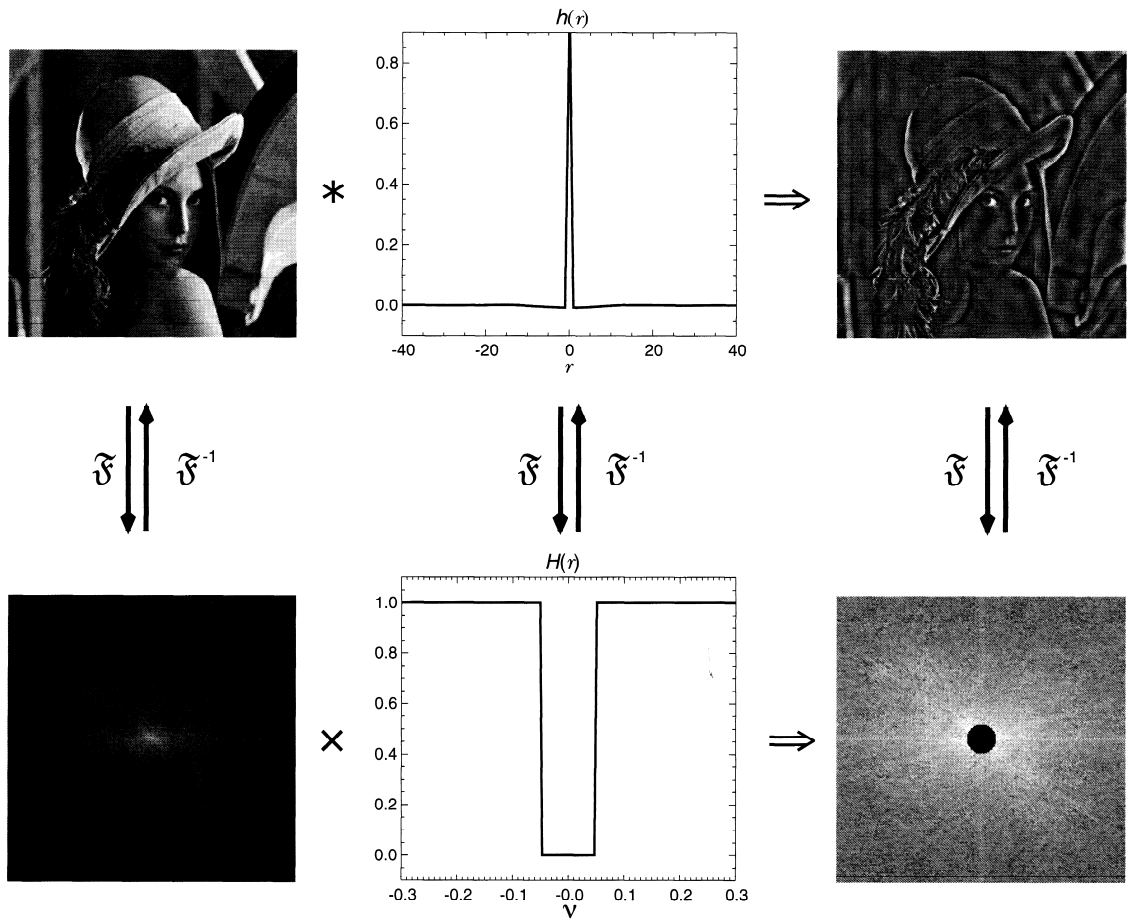
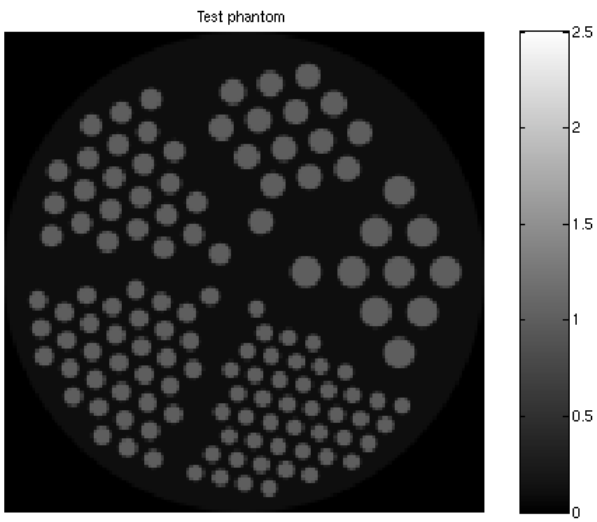
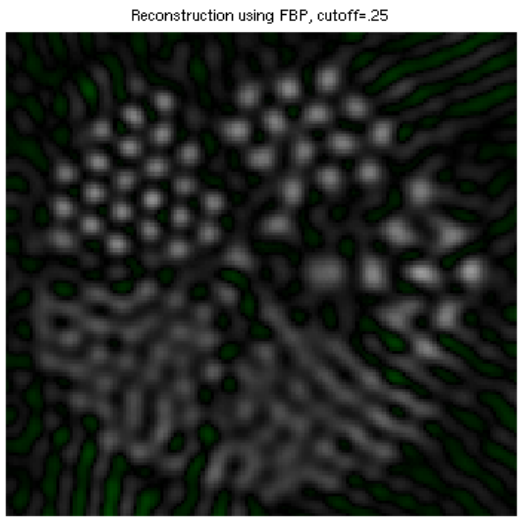


FIGURE 4 Ideal highpass filtering of an image. Figures in the top and bottom rows demonstrate the filtering operation in the spatial and frequency domains, respectively. The left column shows the input image and the magnitude of its Fourier transform. The middle column shows the circular symmetric PSF and its transfer function, where $\nu = \sqrt{\nu_x^2 + \nu_y^2}$ and $r = \sqrt{x^2 + y^2}$. Images in the right column are the output image and the magnitude of its Fourier transform.

Filtered Back-projection

The "ringing" artifacts in reconstructed images.





In this context it is usually manifest itself as "streak artifacts". You may, for example, see this as lines radiating from the center and outwards. The term comes from electronics and is meant in the sense of a bell-ringing.

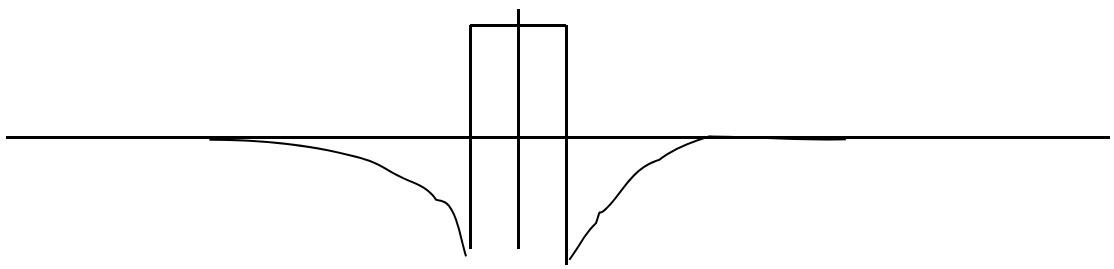
Filtered Back-projection

The Ram-Lak filter in spatial domain

$$h_{\mathrm{RL}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{\mathrm{RL}}(\omega)\exp(ix\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} |\omega| \exp(ix\omega)\, d\omega$$

$$= 2B^2 \operatorname{sinc}(2\pi Bx) - B^2 \operatorname{sinc}^2(\pi Bx)$$



$$\hat{f}(x, y) = \frac{1}{\pi}\int_0^{\pi} d\phi \int_{-\infty}^{\infty} dx'\, p_\phi(x')h(x\cos\phi + y\sin\phi - x')$$

Filtered Back-projection

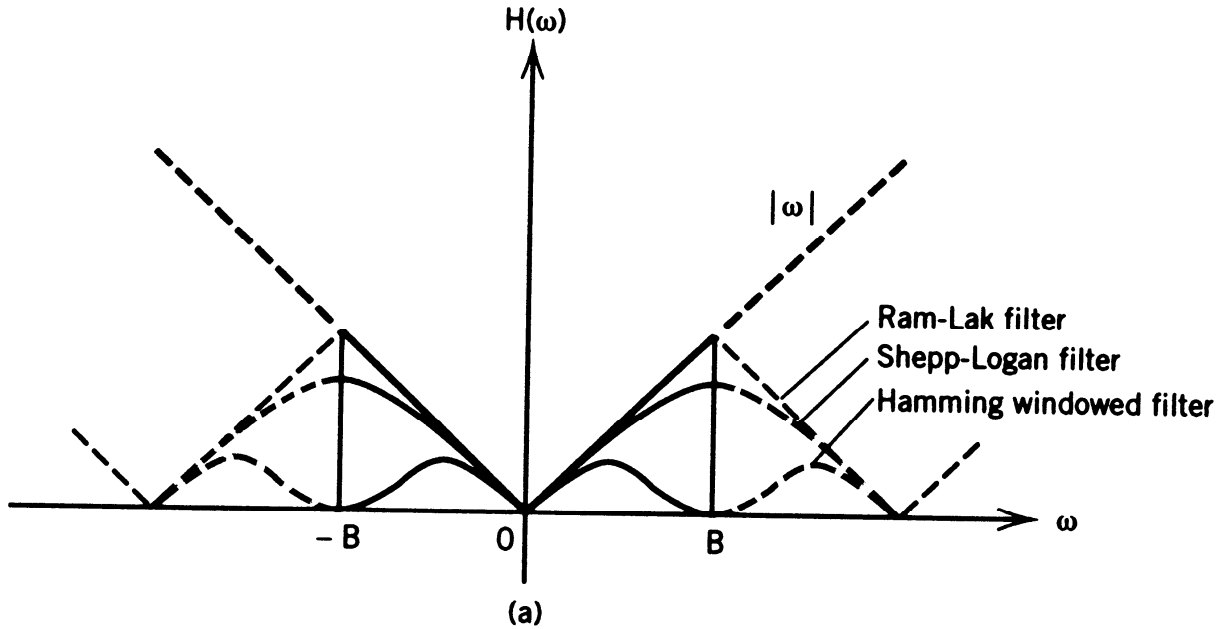


Figure 3-4 (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter.

Filtered Back-Projection

$$\Delta x = 1/(2B),$$

Sampled version

$$h_{\mathrm{RL}}(0) = B^2 = \frac{1}{4\,\Delta x^2} \qquad (\text{if } k = 0)$$

$$h_{\mathrm{RL}}(k) = 0 \qquad (\text{if } k \text{ even})$$

$$h_{\mathrm{RL}}(k) = \frac{-4B^2}{\pi^2 k^2} = \frac{-1}{\pi^2 k^2\,\Delta x^2} \qquad (\text{if } k \text{ odd})$$

$$h_{SL}(k) = \frac{-2}{\pi^2\,\Delta x^2(4k^2 - 1)} = \frac{-8B^2}{\pi^2(4k^2 - 1)}$$

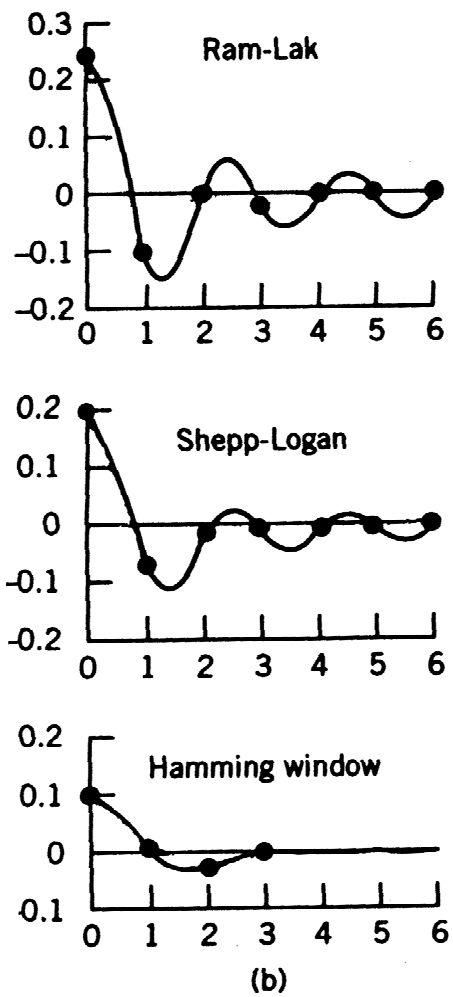


Figure 3-4 (b) Spatial domain filter kernals corresponding to the filter functions shown in the Ram-Lak filter is a high-pass filter with a sharp response but results in some noise enhancement, while the Shepp-Logan and the Hamming window filters are noise-smoothed filters and therefore have better SNR.

Filtered Back-projection

The sharp boundary of the Ram-Lak filter often make the spatial domain filter kernel oscillatory.

It sometime introduces the "ringing" artifacts in reconstructed images.

This can be effectively resolved by the Shepp and Logan filter

$$H_{SL}(\omega) = \begin{cases} |\omega| \sin\left(\dfrac{\omega}{4B}\right), & |\omega| < 2\pi B \\ 0, & \text{otherwise} \end{cases}$$

$$h_{SL}(x) = \frac{B}{\pi^2}\left\{\frac{1 - \cos 2\pi B[(1/4B) + x]}{(1/4B) + x} + \frac{1 - \cos 2\pi B[(1/4B) - x]}{(1/4B) - x}\right\}$$

Filtered Back-projection

- Low spatial frequency data is overweighted. Filter to compensate for this. Weighted by $1/\rho$.

- Solution - filter each projection by $|\rho|$ to account for the uneven sampling density

<u>Steps</u>:

1) Projection operation
2) Transform projection
3) Weight with $|\rho|$
4) Inverse transform
5) Back project
6) Add all angles

Any chance we can define an OPTIMUM filter function for FBP?

Filtered Back-projection

For FBP with noisy projection data, one can derive an optimum filter function. The FBP reconstruction has the minimum mean square error

$$M.S.E = E\left[\left(f(x,y) - \hat{f}(x,y)\right)^2\right]$$

In other words, there exists a filter function that can be used in the FBP reconstruction that produces the most faithful reproduction of the original image f(x,y)

Filtered Back-projection

For a situation in which we know the power spectra for both the image function f(x,y) and the noise, the optimum filter function is

Signal power spectrum of the projection data at view angle ϕ

Where

$$H_{\text{opt}}(\omega,\phi) = |\omega| H_W(\omega,\phi)$$

Image degradation function

$H_w(\omega,\phi)$

$$H_W(\omega,\phi) = \frac{H_D^*(\omega,\phi)\boxed{W_P(\omega,\phi)}}{|\boxed{H_D(\omega,\phi)}|^2 W_P(\omega,\phi) + \boxed{W_{PN}(\omega,\phi)}}$$

Power spectrum of the noise in the projection data at view angle ϕ

Remember that

$$W_p(\omega,\phi) = \left|\mathcal{F}\left[p_\phi(x')\right]\right|^2 = \left\{real\left\{\mathcal{F}\left[p_\phi(x')\right]\right\}\right\}^2 + \left\{imag\left\{\mathcal{F}\left[p_\phi(x')\right]\right\}\right\}^2$$

Fourier Transform

- In general, Fourier transform is a complex valued signal, even if f(x,y) is real valued.
- It is sometimes useful to consider the ***magnitude*** and ***phase*** of the Fourier transform separately.

Fourier coefficients are complex: $$F(u,v) = F_R(u,v) + j \cdot F_I(u,v)$$

Magnitude: $$\left|F(u,v)\right| = \sqrt{F_R^2(u,v) + F_I^2(u,v)}$$

Phase: $$\angle F(u,v) = \tan^{-1}\frac{F_I(u,v)}{F_R(u,v)}$$

An alternative representation: $$F(u,v) = \left|F(u,v)\right| e^{j\angle F(u,v)}$$

- The square of the magnitude $|F(u,v)|^2$ is referred to as the ***power spectrum*** of the original function.

Filtered Back-projection with Optimum Filter

The optimum filter function can also be written as

$$H_W(\omega,\phi) = \frac{H_D^*(\omega,\phi)}{|H_D(\omega,\phi)|^2 + (W_{PN}(\omega,\phi)/W_P(\omega,\phi))}$$

And then

$$H_{\text{opt}}(\omega,\phi) = |\omega| H_W(\omega,\phi)$$

$$= \frac{|\omega| H_D^*(\omega,\phi)}{|H_D(\omega,\phi)|^2 + 1/\text{SNR}(\omega,\phi)}$$

where $[W_P(\omega,\phi)]/[W_{PN}(\omega,\phi)]$ s the signal-to-noise ratio (**SNR**) of the projection data at a given view angle ϕ.

Filtered Back-projection

Therefore the optimum estimator (the optimum reconstruction) of the original image function f(x,y) is

$$\hat{f}(x, y) = \frac{1}{\pi} \int_0^{\pi} d\phi \int_{-\infty}^{\infty} dx' \, \dot{p}_{\phi}^{*}(x') h(x \cos\phi + y \sin\phi - x')$$

where the filtered projection data is given by

$$\mathscr{F}_1^{-1}\left[H_{\text{opt}}(\omega, \phi) \right] * \dot{p}_{\phi}(x') = \dot{p}_{\phi}^{*}(x').$$

Most of current X-ray CT work in fan-beam mode …

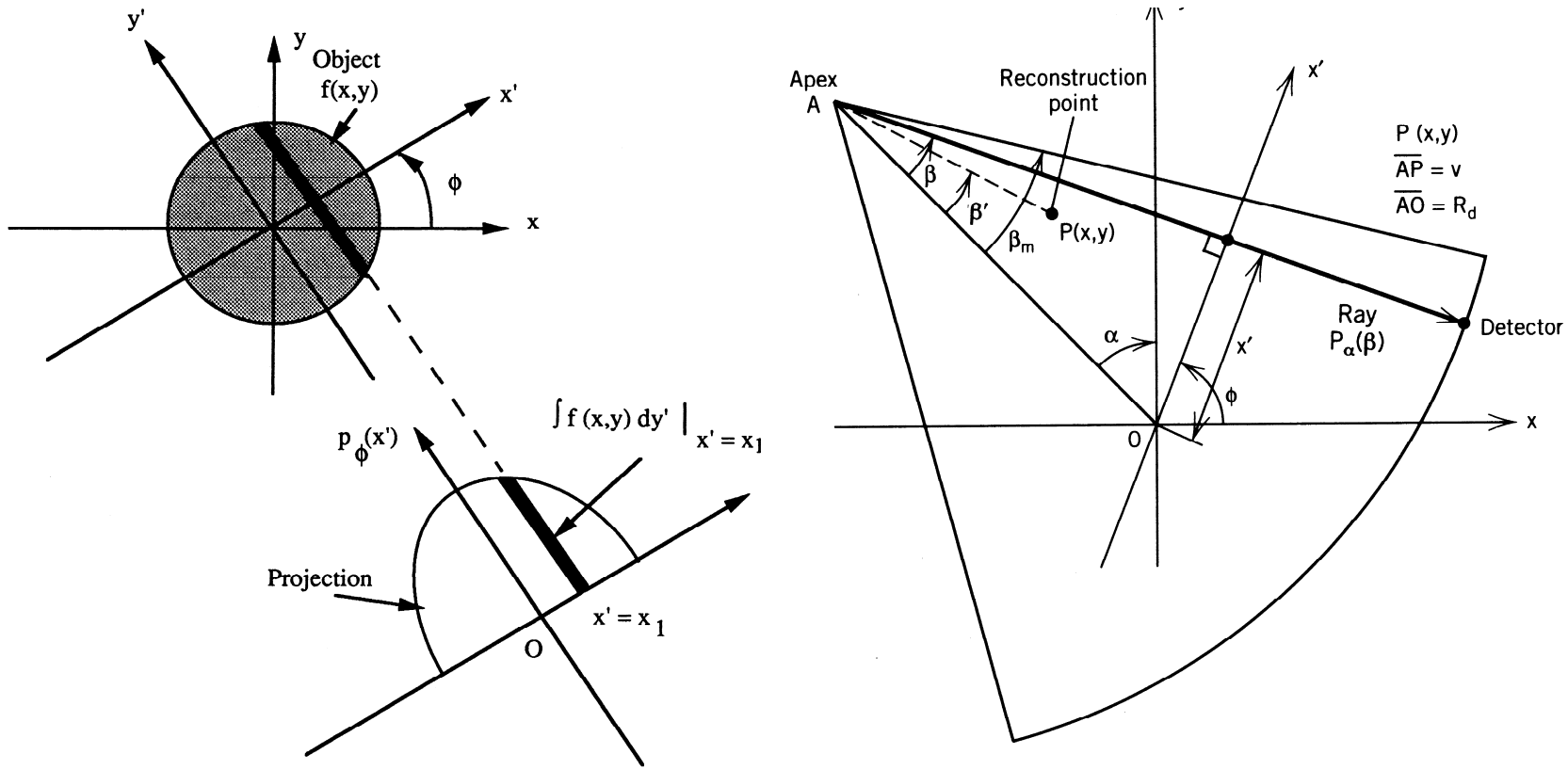
Filtered Back-projection in Fan-beam Mode

Why fan beam mode?

Most modern X-ray CT system use fan (or cone) beam data acquisition scheme.

Image reconstruction with fan beam mode often provide better spatial resolution with the same dimension of sampled data as the parallel case, due to the improved sampling at the central region. This is found to be important for PET, where the intrinsic limitation on spatial resolution is on the finite detector size.

Filtered Back-projection in Fan-beam Mode



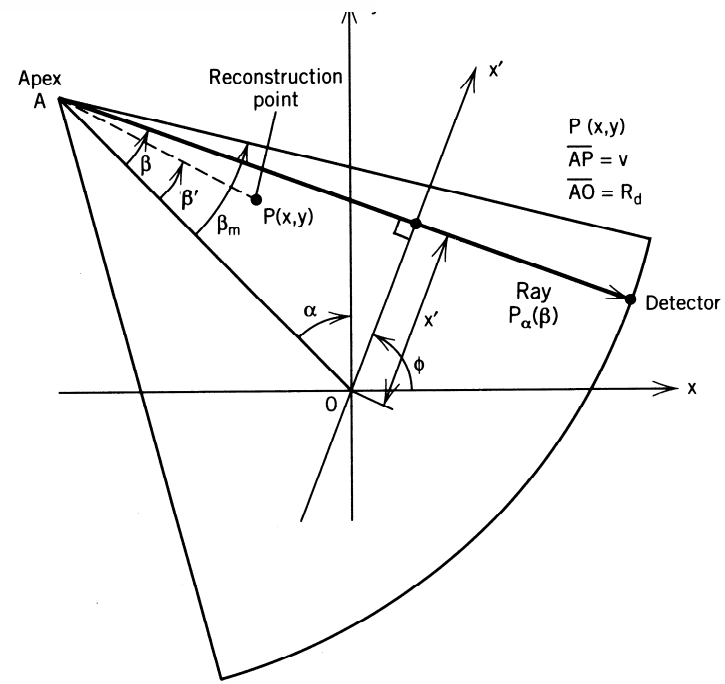
Comparing parallel beam and fan beam geometries

Filtered Back-projection in Fan-beam Mode

where β' is the angle between central line and the line passing through the reconstructed point at (x,y). And v is the distance between the apex of fan and the reconstruction point p.

$$v = \sqrt{(x\cos\alpha + y\sin\alpha)^2 + (x\sin\alpha - y\cos\alpha + R_d)^2}$$

$$\beta' = \tan^{-1}\left[\frac{x\cos\alpha + y\sin\alpha}{x\sin\alpha - y\cos\alpha + R_d}\right]$$



$$x' = R_d \sin\beta$$

$$\phi = \alpha + \beta$$

Filtered Back-projection in Fan-beam Mode

The basic idea:

The mathematical treatment for fan beam mode is almost identical to that for parallel beam case with changing parameters!

Starting from the one-to-one relationship between the parallel beam projection space (ϕ, x') and fan beam projection space (α,β),

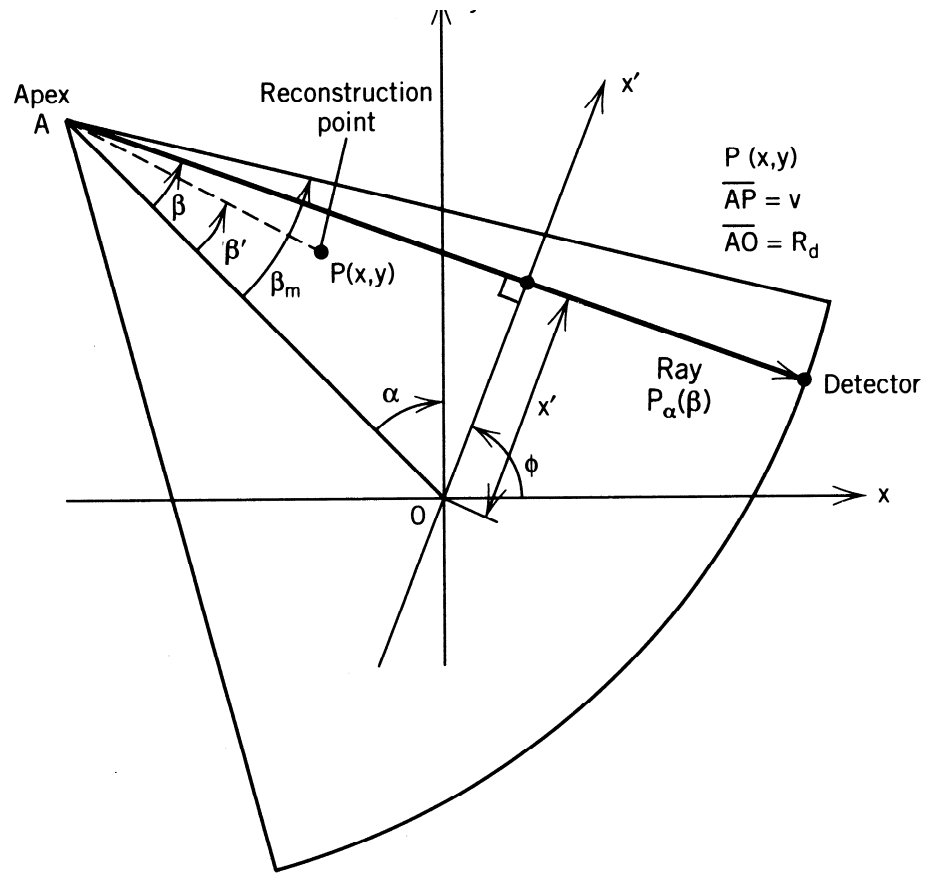
$$x' = R_d \sin \beta$$

$$\phi = \alpha + \beta$$

Filtered Back-projection in Fan-beam Mode

$$x' = R_d \sin \beta$$

$$\phi = \alpha + \beta$$



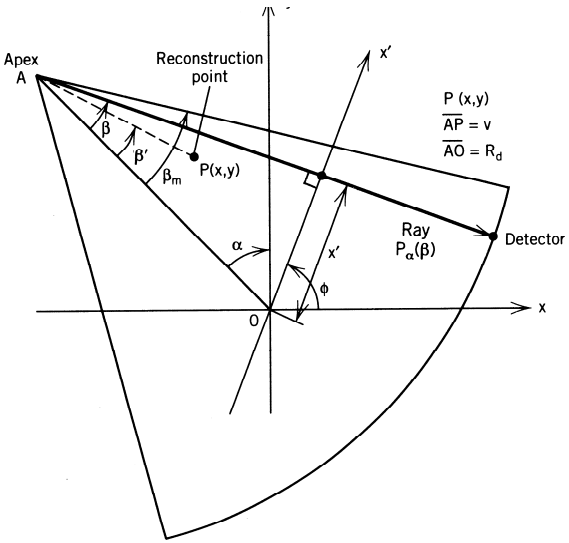
Where the Jacobian |J| is

$$|J| = \left| \frac{\partial(x', \phi)}{\partial(\alpha, \beta)} \right| = R_d \cos \beta$$

Filtered Back-projection in Fan-beam Mode

Where the Jacobian |J| is

$$|J| = \left| \frac{\partial(x', \phi)}{\partial(\alpha, \beta)} \right| = R_d \cos\beta$$



and the FBP in fan beam mode becomes

$$\hat{f}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \int_{-\beta_m}^{\beta_m} d\beta \, p_\alpha(\beta) h\{v \sin(\beta' - \beta)\} |J|$$

Filter Function in Fan-beam Mode

Similarly the filter function used in fan beam mode is a direct transformation from the parallel beam counterpart

$$h(x) = \mathscr{F}_1^{-1}[|\omega|]$$
$$= \mathscr{F}_1^{-1}[H(\omega)]$$

$$x' = R_d \sin\beta$$

$$\phi = \alpha + \beta$$

$$h\{v \sin(\beta' - \beta)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, |\omega| \exp[i\omega v \sin(\beta' - \beta)]$$

Filtered Back-projection in Fan-beam Mode

FBP in parallel beam case

$$\hat{f}(x, y) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dx'\, p_\phi(x')h(x\cos\phi + y\sin\phi - x')$$

$$x' = R_d \sin\beta$$

$$\phi = \alpha + \beta$$

can be converted to FBP in fan beam mode by changing integration variables

$$\hat{f}(x, y) = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \int_{-\beta_m}^{\beta_m} d\beta$$
$$\times p_\alpha(\beta)h\{x\cos(\alpha + \beta) + y\sin(\alpha + \beta) - R_d \sin\beta\}|J|$$

Filtered Back-projection in Fan-beam Mode

Where the Jacibian |J| is

$$|J| = \left| \frac{\partial(x', \phi)}{\partial(\alpha, \beta)} \right| = R_d \cos\beta$$

and the FBP in fan beam mode becomes

$$\hat{f}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \int_{-\beta_m}^{\beta_m} d\beta \, p_\alpha(\beta) h\{v \sin(\beta' - \beta)\} |J|$$