Error and Error Propagation

Two ways to express the error associated with a given measurement:

Probable error:

☞The symmetric range about the mean, within which there is 50% chance that a measurement will fall.

☞The width of the range depends on the distribution of the variable. For example, for Gaussian distributed error, the probable error is $\pm 0.675\ \sigma$.

Fractional standard deviation:

☞The ratio of the standard deviation and the mean of the distribution of the random variable.

☞For Poisson distributed random variable, the fractional standard deviation is simply

$$\frac{\sigma}{\mu} = \frac{1}{\sqrt{\mu}}$$

Error Propagation

In some situations, the variable of interest (Q) is not measured directly, but derived as a function of more than one independent random variable whose values are directly measured. The error on the measured values is propagated into the uncertainty on the resultant quantity Q.

Suppose a quantity Q(x,y) that depends on two independent random variables x and y.

The sample mean and variance of variables x and y are derived as σ_x and σ_y, by repeating measurements.

The standard deviation of the indirect quantity Q is approximately given by

$$\sigma_Q^{\ 2} \cong \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^{\ 2} + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^{\ 2}$$

$$\sigma_Q^{\ 2} \cong \sum_i \left(\frac{\partial Q}{\partial x_i}\right)^2 \sigma_{x_i}^{\ 2}$$

Error and Error Propagation

A Taylor series of a real function of a single variable, f(x), around a point x_0 is given by

$$f(x_0+\Delta x)=f(x_0)+f_x(x_0)\Delta x+\frac{1}{2!}f_{xx}(x_0)\left(\Delta x\right)^2+\frac{1}{3!}f_{xxx}(x_0)\left(\Delta x\right)^3+\ldots\ldots$$

where

$$f_{xx}(x_0)=\left[\frac{d}{dx}\frac{d}{dx}f(x)\right]\Bigg|_{x=x_0}$$

A Taylor series of a real function of two variables, f(x,y), is given by

$$f(x_0+\Delta x,y_0+\Delta y)=$$
$$f(x_0,y_0)+\left[f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y\right]$$
$$+\frac{1}{2!}\left[f_{xx}(x_0,y_0)(\Delta x)^2+2f_{xy}(x_0,y_0)\Delta x\Delta y+f_{yy}(x_0,y_0)(\Delta y)^2\right]$$
$$+\frac{1}{3!}\left[f_{xxx}(x_0,y_0)(\Delta x)^3+3f_{xxy}(x_0,y_0)(\Delta x)^2(\Delta y)+3f_{xyy}(x_0,y_0)(\Delta x)(\Delta y)^2+f_{yyy}(x_0,y_0)(\Delta y)^3\right]+\ldots\ldots$$

Error Propagation

We determine the standard deviation of a quantity $Q(x, y)$ that depends on two independent, random variables x and y. A sample of N measurements of the variables yields pairs of values, x_i and y_i, with $i = 1, 2, \ldots, N$. For the sample one can compute the means, $\bar{x}$ and $\bar{y}$; the standard deviations, σ_x and σ_y; and the values $Q_i = Q(x_i, y_i)$. We assume that the scatter of the x_i and y_i about their means is small. We can then write a power-series expansion for the Q_i about the point $(\bar{x}, \bar{y})$, keeping only the first powers. Thus,

$$Q_i = Q(x_i, y_i) \cong Q(\bar{x}, \bar{y}) + \frac{\partial Q}{\partial x}(x_i - \bar{x}) + \frac{\partial Q}{\partial y}(y_i - \bar{y}), \qquad \text{(E.36)}$$

where the partial derivatives are evaluated at $x = \bar{x}$ and $y = \bar{y}$.

So the mean of Q is

$$\bar{Q} = \frac{1}{N}\sum_{i=1}^{N} Q_i = \frac{1}{N}\sum_{i=1}^{N} Q(x_i, y_i)$$

and the variance of Q is

$$\sigma^2(Q) = \frac{1}{N}\sum_{i=1}^{N}\left(Q_i - \bar{Q}\right)^2$$

Error Propagation

where the partial derivatives are evaluated at $x = \bar{x}$ and $y = \bar{y}$. The mean value of Q_i is simply

$$\bar{Q} \equiv \frac{1}{N}\sum_{i=i}^{N} Q_i = \frac{1}{N}\sum_{i=1}^{N} Q(x_i, y_i)$$

$$\cong \frac{1}{N}\sum_{i=1}^{N}\left[Q(\bar{x},\bar{y}) + \left.\frac{\partial Q}{\partial x}\right|_{(\bar{x},\bar{y})}(x_i - \bar{x}) + \left.\frac{\partial Q}{\partial y}\right|_{(\bar{x},\bar{y})}(y_i - \bar{y})\right] = Q(\bar{x},\bar{y}), \quad \text{E.36}$$

since the sums of the $x_i - \bar{x}$ and $y_i - \bar{y}$ over all i in Eq. (E.36) are zero, by definition of the mean values. Thus, the mean value of Q is the value of the function $Q(x, y)$ calculated at $x = \bar{x}$ and $y = \bar{y}$.

Error and Error Propagation

The variance of the Q_i is given by

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}(Q_i - \overline{Q})^2. \qquad \text{(E.38)}$$

$$Q_i = Q(x_i, y_i) \cong Q(\bar{x}, \bar{y}) + \frac{\partial Q}{\partial x}(x_i - \bar{x}) + \frac{\partial Q}{\partial y}(y_i - \bar{y}), \qquad \text{(E.36)}$$

Applying Eq. (E.36) with $\overline{Q} = Q(\bar{x}, \bar{y})$, we find that

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial Q}{\partial x}(x_i - \bar{x}) + \frac{\partial Q}{\partial y}(y_i - \bar{y})\right]^2 \qquad \text{(E.39)}$$

$$= \left(\frac{\partial Q}{\partial x}\right)^2 \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{y})^2$$

$$+ 2\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial Q}{\partial y}\right)\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y}). \qquad \text{(E.40)}$$

Variance of y, or $\sigma^2(y)$

Covariance, Cov(x, y)

Error and Error Propagation

The last term, called the *covariance* of x and y, vanishes for large N if the values of x and y are uncorrelated. (The factors $y_i - \bar{y}$ and $x_i - \bar{x}$ are then just as likely to be positive as negative, and the covariance also decreases as $1/N$). We are left with the first two terms, involving the variances of the x_i and y_i:

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^2. \qquad \text{(E.41)}$$

This is one form of the error propagation formula, which is easily generalized to a function Q of any number of independent random variables.

Assumptions ??

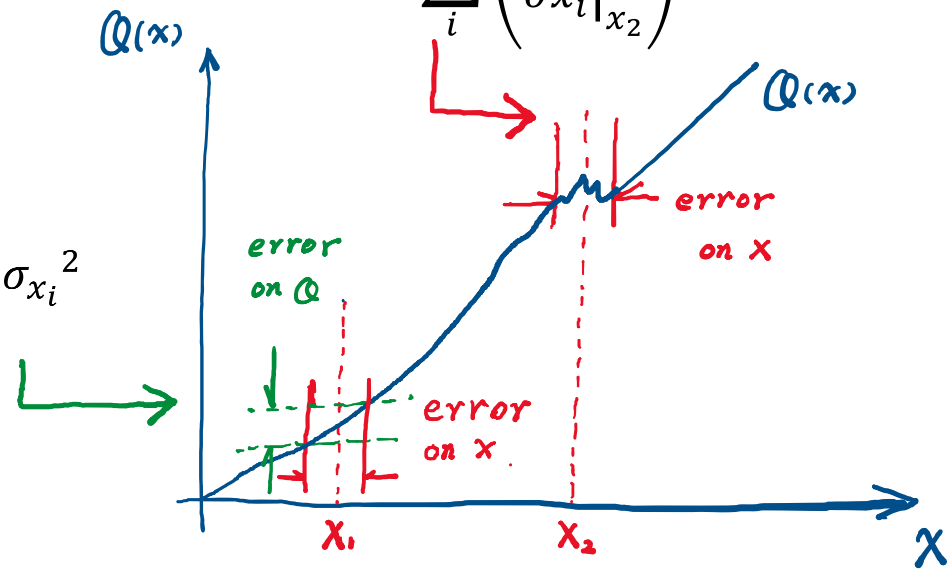
Error Propagation Formula

The error propagation formula is exact only when

- the two variables, x and y, are independent to each other,
- and when Q(x,y) could be approximated as a linear function of both x and y.

✗ $$\sigma_Q{}^2 \cong \sum_i \left(\frac{\partial Q}{\partial x_i}\bigg|_{x_2} \right)^2 \sigma_{x_i}{}^2$$

✓ $$\sigma_Q{}^2 \cong \sum_i \left(\frac{\partial Q}{\partial x_i}\bigg|_{x_1} \right)^2 \sigma_{x_i}{}^2$$



Error Propagation Formula

The error propagation formula is exact only when

- the two variables, x and y, are independent to each other,
- and when Q(x,y) could be approximated as a linear function of both x and y.

Note that the formula would break down when the second and third and higher order partial derivatives are not negligible.

$$
\begin{aligned}
&f(x_0+\Delta x, y_0+\Delta y) = \\
&f(x_0,y_0) + \left[f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y\right] \\
&+\frac{1}{2!}\left[f_{xx}(x_0,y_0)(\Delta x)^2 + 2f_{xy}(x_0,y_0)\Delta x\Delta y + f_{yy}(x_0,y_0)(\Delta y)^2\right] \\
&+\frac{1}{3!}\left[f_{xxx}(x_0,y_0)(\Delta x)^3 + 3f_{xxy}(x_0,y_0)(\Delta x)^2(\Delta y) + 3f_{xyy}(x_0,y_0)(\Delta x)(\Delta y)^2 + f_{yyy}(x_0,y_0)(\Delta y)^3\right] + \ldots\ldots
\end{aligned}
$$

Error Propagation

Case 1: Sums or differences of counts – u is the sum or difference of two random numbers representing counts measured in two independent experiments.

$$u = x + y \quad \text{or} \quad u = x - y$$

$$\sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2}$$

$$\sigma_Q{}^2 \cong \sum_i \left(\frac{\partial Q}{\partial x_i}\right)^2 \sigma_{x_i}{}^2 ,$$

or

$$\sigma_Q \cong \sqrt{\sum_i \left(\frac{\partial Q}{\partial x_i}\right)^2 \sigma_{x_i}{}^2} .$$

Example: estimating the net counts from a sample.

$$\text{net counts} = \text{total counts} - \text{background counts}$$

or

$$u = x - y$$

Error Propagation

Case 2: Multiplication or division by a constant

$$u = Ax$$

$$\Downarrow$$

$$\sigma_u = A\sigma_x$$

Example: estimating the count rate, $\text{counting rate} \equiv r = \dfrac{x}{t}$

Assuming that the error in the measuring time is negligible, we get

$$\sigma_r = \frac{\sigma_x}{t}$$

Error Propagation

Case 3: Multiplication or division of counts

$$u = xy, \quad \frac{\partial u}{\partial x} = y \quad \frac{\partial u}{\partial y} = x$$

Using the equation

$$\sigma_Q{}^2 \cong \sum_i \left(\frac{\partial Q}{\partial x_i} \right)^2 \sigma_{x_i}{}^2$$

One gets

$$\sigma_u^2 = \frac{\partial u}{\partial x} \sigma_x^2 + \frac{\partial u}{\partial y} \sigma_y^2 = y \cdot \sigma_x^2 + x \cdot \sigma_y^2 \, .$$

Therefore,

$$\boxed{\left(\frac{\sigma_u}{u} \right)^2 = \left(\frac{\sigma_x}{x} \right)^2 + \left(\frac{\sigma_y}{y} \right)^2}$$

Error Propagation in Net Count Rate Measurement

As an application of the error-propagation formula, Eq. (11.46), we find the standard deviation of the net count rate of a sample, obtained experimentally as the difference between gross and background count rates, r_g and r_b. As with gross counting, one also measures the number n_b of background counts in a time t_b. The net count rate ascribed to the sample is then the difference

$$r_n = r_g - r_b = \frac{n_g}{t_g} - \frac{n_b}{t_b}. \qquad \sigma_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^2. \tag{11.49}$$

To find the standard deviation of r_n, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $y = n_b$. From Eq. (11.49) we have $\partial r_n/\partial n_g = 1/t_g$ and $\partial r_n/\partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

Assuming no error on t

$$\sigma_{nr} = \sqrt{\frac{\sigma_g^2}{t_g^2} + \frac{\sigma_b^2}{t_b^2}} = \sqrt{\sigma_{gr}^2 + \sigma_{br}^2}. \tag{11.50}$$

where $\sigma_{nr}^2 \equiv \sigma^2(r_n), \quad \sigma_g^2 \equiv \sigma^2(n_g), \quad \sigma_g^2 \equiv \sigma^2(n_b),$

and $\quad \sigma_{gr}^2 = \sigma^2\left(\frac{n_g}{t_g}\right) = \frac{\sigma_g^2}{t_g^2}, \quad \sigma_{br}^2 = \sigma^2\left(\frac{n_b}{t_b}\right) = \frac{\sigma_b^2}{t_b^2}.$

Turner, pp. 324.

Error Propagation in Net Count Rate Measurement

To find the standard deviation of r_n, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $y = n_b$. From Eq. (11.49) we have $\partial r_n/\partial n_g = l/t_g$ and $\partial r_n/\partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

$$\sigma_{nr} = \sqrt{\frac{\sigma_g^2}{t_g^2} + \frac{\sigma_b^2}{t_b^2}} = \sqrt{\sigma_{gr}^2 + \sigma_{br}^2}. \quad (11.50)$$

Here σ_g and σ_b are the standard deviations of the numbers of gross and background counts, and σ_{gr} and σ_{br} are the standard deviations of the gross and background count rates. Equation (11.50) expresses the well-known result for the standard deviation of the sum or difference of two Poisson or normally distributed random variables. Using n_g and n_b as the best estimates of the means of the gross and background distributions and assuming that the numbers of counts obey Poisson statistics, we have $\sigma_g^2 = n_g$ and $\sigma_b^2 = n_b$. Therefore, the last equation can be written

$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}}, \quad (11.51)$$

Error Propagation in Net Count Rate Measurement

As an application of the error-propagation formula, Eq. (11.46), we find the standard deviation of the net count rate of a sample, obtained experimentally as the difference between gross and background count rates, r_g and r_b. As with gross counting, one also measures the number n_b of background counts in a time t_b. The net count rate ascribed to the sample is then the difference

$$r_n = r_g - r_b = \frac{n_g}{t_g} - \frac{n_b}{t_b}. \qquad \sigma_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^2. \qquad (11.49)$$

To find the standard deviation of r_n, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $y = n_b$. From Eq. (11.49) we have $\partial r_n / \partial n_g = 1/t_g$ and $\partial r_n / \partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

Assuming no error on t

$$\sigma_{nr} = \sqrt{\frac{\sigma_g^2}{t_g^2} + \frac{\sigma_b^2}{t_b^2}} = \sqrt{\sigma_{gr}^2 + \sigma_{br}^2}. \qquad (11.50)$$

or

$$\sigma_{nr} \cong \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} \cong \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}},$$

Turner, pp. 324.

Error Propagation in Net Count Rate Measurement

Example

A long-lived radioactive sample is placed in a counter for 10 min, and 1426 counts are registered. The sample is then removed, and 2561 background counts are observed in 90 min. (a) What is the net count rate of the sample and its standard deviation? (b) If the counter efficiency with the sample present is 28%, what is the activity of the sample and its standard deviation in Bq? (c) Without repeating the background measurement, how long would the sample have to be counted in order to obtain the net count rate to within ±5% of its true value with 95% confidence? (d) Would the time in (c) also be sufficient to ensure that the *activity* is known to within ±5% with 95% confidence?

Turner, pp. 324.

Error Propagation in Net Count Rate Measurement

(a) What is the net count rate of the sample and its standard deviation?

Solution

(a) We have $n_g = 1426$, $t_g = 10$ min, $n_b = 2561$, and $t_b = 90$ min. The gross and background count rates are $r_g = 1426/10 = 142.6$ cpm and $r_b = 2561/90 = 28.5$ cpm.

Therefore, the net count rate is $r_n = 142.6 - 28.5 = 114$ cpm. The standard deviation can be found from either of the expressions in (11.51). Using the first (which does not depend on the calculated values, r_g and r_b), we find

$$\sigma_{nr} = \sqrt{\frac{1426}{(10\ \text{min})^2} + \frac{2561}{(90\ \text{min})^2}} = 3.82\ \text{min}^{-1} = 3.82\ \text{cpm}. \qquad (11.52)$$

$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}},$$

Error Propagation in Net Count Rate Measurement

(b) If the counter efficiency with the sample present is 28%, what is the activity of the sample and its standard deviation in Bq? (c) Without repeat ; the background

Solution:

(b) Since the counter efficiency is $\epsilon = 0.28$, the inferred activity of the sample is $A = r_n/\epsilon = (114\ \text{min}^{-1})/0.28 = 407$ dpm $= 6.78$ Bq. The standard deviation of the activity is $\sigma_{nr}/\epsilon = (3.82\ \text{min}^{-1})/0.28 = 13.6$ dpm $= 0.227$ Bq.

$$u = Ax$$

$$\sigma_u = A\sigma_x$$

Error Propagation in Net Count Rate Measurement

(c) Without repeating the background measurement, how long would the sample have to be counted in order to obtain the net count rate to within ±5% of its true value with 95% confidence? (d) Would the time in (c) also be sufficient to ensure that the *activity* is known to within ±5% with 95% confidence?

Solution:

(c) A 5% uncertainty in the net count rate is $0.05r_n = 0.05 \times 114 = 5.70$ cpm. For the true net count rate to be within this range of the mean at the 95% confidence level means that 5.70 cpm $= 1.96\sigma_{nr}$ (Table 11.2), or that $\sigma_{nr} = 2.91$ cpm. Using the second expression in (11.51) with the background rate as before (since we do not yet know the new value of n_g), we write

$$u = Ax$$

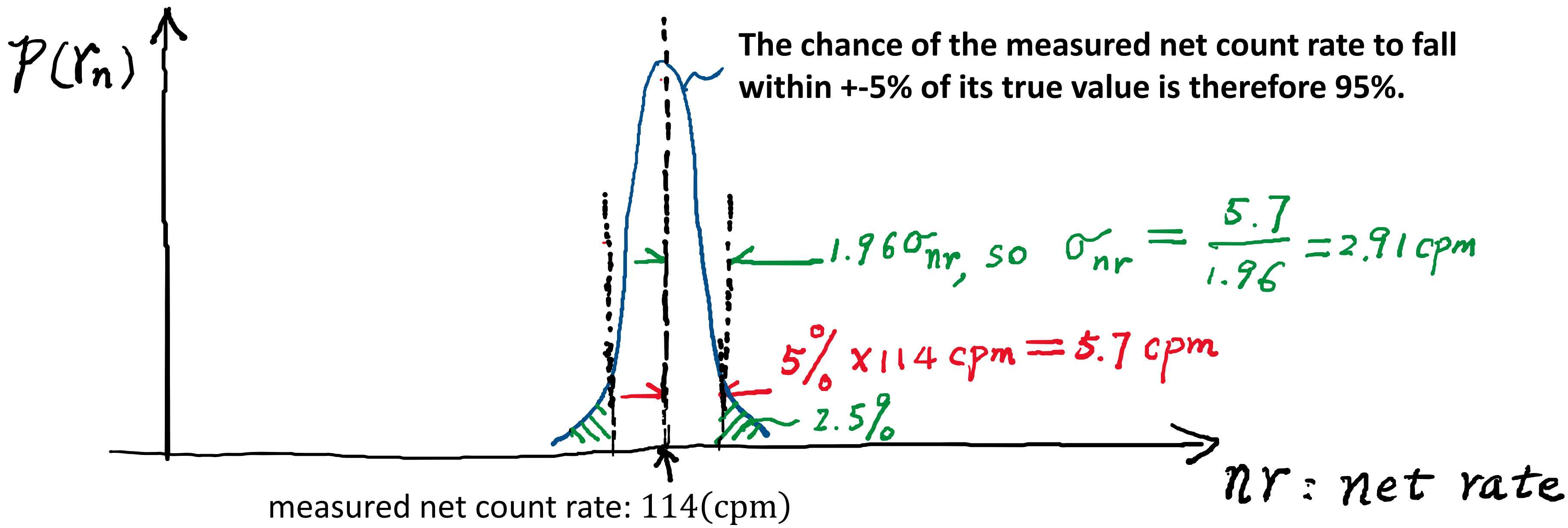
$$\sigma_{nr} = 2.91 \text{ min}^{-1} = \sqrt{\frac{142.6 \text{ min}^{-1}}{t_g} + \frac{28.5 \text{ min}^{-1}}{90 \text{ min}}}. \quad (11.53)$$

$$\sigma_u = A\sigma_x$$

Solving, we find that $t_g = 17.5$ min.

(d) Yes. The relative uncertainties remain the same and scale according to the efficiency. If the efficiency were larger and the counting times remained the same, then a larger number of counts and less statistical uncertainty would result.

Error Propagation in Net Count Rate Measurement



If we assume that the measured net count rate of 114 cpm is close enough to the true net count rate, then to ensure there is 95% chance that the measured net count rate would fall within +-5% of its true value, we need

$$114(cpm) \times 5\% = 1.96 \times \sigma_{nr}.$$

Remember that

$$\sigma_{\mathrm{nr}} = \sqrt{\frac{n_{\mathrm{g}}}{t_{\mathrm{g}}^2} + \frac{n_{\mathrm{b}}}{t_{\mathrm{b}}^2}} = \sqrt{\frac{r_{\mathrm{g}}}{t_{\mathrm{g}}} + \frac{r_{\mathrm{b}}}{t_{\mathrm{b}}}},$$

then

$$5.71(cpm) = 1.96 \cdot \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}} = 1.96 \sqrt{\frac{r_b + 114(cpm)}{t_{\boldsymbol{g}}} + \frac{r_b}{t_b}}, \qquad \boldsymbol{so}\ \ t_g = 17.5\ (cpm)$$

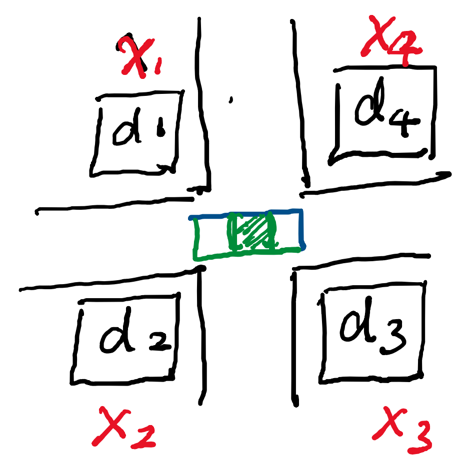
Error Propagation

Case 5: Combination of independent measurements with unequal errors

If N independent measurements of the same quantity have been carried out and not all the measurements have the same precision, what is the best way to estimate the best estimate of the mean value of the quantity to be measured?

The best estimate of the quantity, <x>, can be achieved by the weighted average

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$



How to assign the weighting factors a_i's ?

Error Propagation

Let each individual measurement x_i be given a weighting factor a_i and the best value $\langle x \rangle$ computed from the linear combination

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i} \tag{3.45}$$

We now seek a criterion by which the weighting factors a_i should be chosen in order to minimize the expected error in $\langle x \rangle$.

For brevity, we write

$$\alpha \equiv \sum_{i=1}^{N} a_i$$

so that

$$\langle x \rangle = \frac{1}{\alpha} \sum_{i=1}^{N} a_i x_i$$

$$\sigma_Q{}^2 \cong \sum_i \left(\frac{\partial Q}{\partial x_i}\right)^2 \sigma_{x_i}{}^2 .$$

Now apply the error propagation formula [Eq. (3.37)] to this case:

$$\sigma_{\langle x \rangle}^2 = \sum_{i=1}^{N} \left(\frac{\partial \langle x \rangle}{\partial x_i}\right)^2 \sigma_{x_i}^2$$

Knoll, p. 91.

Error Propagation

Now apply the error propagation formula [Eq. (3.37)] to this case:

$$\sigma_{\langle x\rangle}^2 = \sum_{i=1}^{N} \left(\frac{\partial\langle x\rangle}{\partial x_i}\right)^2 \sigma_{x_i}^2$$

$$\langle x\rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$

$$= \sum_{i=1}^{N} \left(\frac{a_i}{\alpha}\right)^2 \sigma_{x_i}^2$$

$$= \frac{1}{\alpha^2} \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2$$

$$\sigma_{\langle x\rangle}^2 = \frac{\beta}{\alpha^2} \qquad (3.46)$$

where

$$\alpha \equiv \sum_{i=1}^{N} a_i \qquad \beta \equiv \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2$$

In order to minimize $\sigma_{\langle x\rangle}$, we must minimize $\sigma_{\langle x\rangle}^2$ from Eq. (3.46) with respect to a typical weighting factor a_j:

$$0 = \frac{\partial \sigma_{\langle x\rangle}^2}{\partial a_j} = \frac{\alpha^2 \dfrac{\partial\beta}{\partial a_j} - 2\alpha\beta \dfrac{\partial\alpha}{\partial a_j}}{\alpha^4} \qquad (3.47)$$

Error Propagation

$$0 = \frac{\partial \sigma^2_{\langle x \rangle}}{\partial a_j} = \frac{\alpha^2 \dfrac{\partial \beta}{\partial a_j} - 2\alpha\beta \dfrac{\partial \alpha}{\partial a_j}}{\alpha^4} \tag{3.47}$$

Note that

$$\alpha \equiv \sum_{i=1}^{N} a_i \quad \longrightarrow \quad \frac{\partial \alpha}{\partial a_j} = 1 \qquad \frac{\partial \beta}{\partial a_j} = 2a_j\sigma^2_{x_j} \quad \longleftarrow \quad \beta \equiv \sum_{i=1}^{N} a_i^2 \sigma^2_{x_i}$$

Putting these results into Eq. (3.47), we obtain

$$\frac{1}{\alpha^4}(2\alpha^2 a_j \sigma^2_{x_j} - 2\alpha\beta) = 0$$

and solving for a_j, we find

$$a_j = \frac{\beta}{\alpha} \cdot \frac{1}{\sigma^2_{x_j}} \tag{3.48}$$

If we choose to normalize the weighting coefficients,

$$\sum_{i=1}^{N} a_i \equiv \alpha = 1$$

$$a_j = \frac{\beta}{\sigma^2_{x_j}}$$

Error Propagation

$$a_j = \frac{\beta}{\sigma_{x_j}^2}$$

Putting this into the definition of β, we obtain

$$\beta = \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2 = \sum_{i=1}^{N} \left(\frac{\beta}{\sigma_{x_i}^2} \right)^2 \sigma_{x_i}^2$$

or

$$\beta = \left(\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1} \qquad (3.49)$$

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$

Therefore, the proper choice for the normalized weighting coefficient for x_j, is

$$\boxed{a_j = \frac{1}{\sigma_{x_j}^2} \left(\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1}} \qquad (3.50)$$

We therefore see that *each data point should be weighted inversely as the square of its own error.*

Error Propagation

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$

Therefore, the proper choice for the normalized weighting coefficient for x_j, is

$$a_j = \frac{1}{\sigma_{x_j}^2}\left(\sum_{i=1}^{N}\frac{1}{\sigma_{x_i}^2}\right)^{-1} \qquad (3.50)$$

We therefore see that *each data point should be weighted inversely as the square of its own error.*

$$\frac{1}{\sigma_{x_i}^2} \sim \text{creditability of the measurement } x_i.$$

$$a_i = \frac{1}{\sigma_{x_i}^2} \Big/ \left(\sum_{i=1}^{N}\frac{1}{\sigma_{x_i}^2}\right) \sim \text{relative } \textbf{creditability} \text{ of the measurement } x_i.$$

Error Propagation

Assuming that this optimal weighting is followed, what will be the resultant (minimum) error in $\langle x \rangle$? Because we have chosen $\alpha = 1$ for normalization, Eq. (3.46) becomes

$$\sigma_{\langle x \rangle}^2 = \beta$$

In the case of optimal weighting, β is given by Eq. (3.49). Therefore,

$$\boxed{\frac{1}{\sigma_{\langle x \rangle}^2} = \sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2}} \tag{3.51}$$

From Eq. (3.51), the expected standard deviation $\sigma_{\langle x \rangle}$ can be calculated from the standard deviations σ_{x_i} associated with each individual measurement.

Error Propagation

Case 5: Combination of independent measurements with unequal errors (continued)

The proper choice for the normalized weighting factors for x_i is

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$

$$a_j = \frac{1}{\sigma_{x_j}^2}\left(\sum_{i=1}^{N}\frac{1}{\sigma_{x_i}^2}\right)^{-1}$$

And the error (variance) on the weighted average is

$$\sigma_{\langle x \rangle}^2 = \left(\sum_{i=1}^{N}\frac{1}{\sigma_{x_i}^2}\right)^{-1}$$

Optimization of Counting Experiments

Case 6: Measuring the net count rate from a long-lived radioisotope.

$S \equiv$ counting rate due to the source alone without background

$B \equiv$ counting rate due to background

The measurement of S is normally carried out by counting the source plus background (at an average rate of $S + B$) for a time T_{S+B} and then counting background alone for a time T_B. The net rate due to the source alone is then

$$S = \frac{N_1}{T_{S+B}} - \frac{N_2}{T_B} \qquad 2)$$

where N_1 and N_2 are the total counts in each measurement.

If the total measurement $T=T_{S+B}+T_B$ is fixed, how to minimize the statistical error on the measured net count rate?

Error Propagation in Net Count Rate Measurement

To find the standard deviation of r_n, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $y = n_b$. From Eq. (11.49) we have $\partial r_n / \partial n_g = l/t_g$ and $\partial r_n / \partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

$$\sigma_{nr} = \sqrt{\frac{\sigma_g^2}{t_g^2} + \frac{\sigma_b^2}{t_b^2}} = \sqrt{\sigma_{gr}^2 + \sigma_{br}^2}. \tag{11.50}$$

Here σ_g and σ_b are the standard deviations of the numbers of gross and background counts, and σ_{gr} and σ_{br} are the standard deviations of the gross and background count rates. Equation (11.50) expresses the well-known result for the standard deviation of the sum or difference of two Poisson or normally distributed random variables. Using n_g and n_b as the best estimates of the means of the gross and background distributions and assuming that the numbers of counts obey Poisson statistics, we have $\sigma_g^2 = n_g$ and $\sigma_b^2 = n_b$. Therefore, the last equation can be written

$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}}, \tag{11.51}$$

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Applying the results of error propagation analysis to Eq. (3.52), we obtain

$$\sigma_S = \left[\left(\frac{\sigma_{N_1}}{T_{S+B}}\right)^2 + \left(\frac{\sigma_{N_2}}{T_B}\right)^2\right]^{1/2}$$

$$\sigma_S = \left(\frac{N_1}{T_{S+B}^2} + \frac{N_2}{T_B^2}\right)^{1/2}$$

$$\sigma_S = \left(\frac{S+B}{T_{S+B}} + \frac{B}{T_B}\right)^{1/2}$$

N_1: measured counts during the source+background measurement.

N_2: measured counts during the background-only measurement.

S: measured **count-rate** during the source+background measurement.

B: measured **count-rate** during the background - only measurement.

If we now assume that a fixed total time $T = T_{S+B} + T_B$ is available to carry out both measurements, the above uncertainty can be minimized by optimally choosing the fraction of T allocated to T_{S+B} (or T_B). We square Eq. (3.53) and differentiate

$$2\sigma_S\, d\sigma_S = -\frac{S+B}{T_{S+B}^2}\, dT_{S+B} - \frac{B}{T_B^2}\, dT_B = 0$$

and set $d\sigma_S = 0$ to find the optimum condition. Also, because T is a constant, $dT_{S+B} + dT_B = 0$. The optimum division of time is then obtained by meeting the condition

$$\left.\frac{T_{S+B}}{T_B}\right|_{\text{opt}} = \sqrt{\frac{S+B}{B}} \qquad (3.54)$$